University of Washington
Department of Civil and Environmental Engineering

OPERATIONAL COMPARISON OF STOCHASTIC STREAMFLOW GENERATON PROCEDURES

Stephen J. Burges
Dennis P. Lettenmaier

Water Resources Series
Technical Report No. 45
November 1975

Seattle, Washington
98195

Department of Civil Engineering
University of Washington
Seattle, Washington 98195

OPERATIONAL COMPARISON OF STOCHASTIC STREAMFLOW GENERATON PROCEDURES

Stephen J. Burges
Dennis P. Lettenmaier

Water Resources Series
Technical Report No. 45

November 1975

Charles W. Harris Hydraulics Laboratory

Department of Civil Engineering

University of Washington

Seattle, Washington 98195

OPERATIONAL COMPARISON OF STOCHASTIC STREAMFLOW GENERATION PROCEDURES

by

Stephen J. Burges and Dennis P. Lettenmaier

Technical Report No. 45

November 1975

Project Completion Report:	"Operational Comparison of Stochastic Streamflow Generation Procedures"
Project Period:	January 1, 1975 to June 30, 1975
Principal Investigator:	Stephen J. Burges, Associate Professor of Civil Engineering, University of Washington
OWRT Project Number:	A-073-WASH
OWRT Agreement Number:	14-31-0001-5084/3034

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

LIST OF TABLES

LIST OF TABLES (CONT.)

LIST OF FIGURES

LIST OF FIGURES (CONT.)

LIST OF FIGURES (CONT.)

FIGURE PAGE

Acknowledgements

The authors wish to thank J. R. Slack and M. Moss (USGS-Reston) for their timely advice at the beginning of this investigation. The expedious delivery of recent Ph.D. theses from P. E. O'Connell and U. Jettmar helped the direction of the research effort. The good humor and cooperation of the staff of the University of Washington Computer Center contributed to the authors' efforts at crucial times. The cheerful cooperation of the secretaries in the Water and Air Resources Division, Department of Civil Engineering is appreciated; Mrs. Leslie Caldwell typed the manuscript.

The research was supported by a grant from the Office of Water Research and Technology administered by the State of Washington Water Research Center.

ABSTRACT

Operational comparisons are presented using the sequent Peak Algorithm to estimate probability storage distributions resulting from use of short and long memory synthetic annual streamflow generators at a single site. The short memory model used was the Lag one Markov process, while a Fast Fractional Gaussian Noise (FFGN) generator was used to generate self-similar (long memory) sequences. In addition, a simpler model of long term effects, the ARMA(1,1) process, was compared to the FFGN generator. In all cases, 1000 sequences each of length 40 were used to generate the empirical probability storage distributions, which were approximated using an Extreme Value Type I distribution. Skewed marginal distributions were modelled using a three parameter log normal distribution. The model parameters (taken as population values) which were investigated were coefficient of variation, lag one correlation coefficient, skew coefficient, and (for self-similar models) Hurst coefficient. For each trace, the storages required to satisfy constant annual demand to mean annual flow ratios of 0.5, 0.7, and 0.9 were derived.

The results showed that modelling of the Hurst effect alone is insufficient to gaurantee accurate results. Even at relatively large H values, correct modelling of the lag one correlation coefficient was found to be quite important. In addition, skew coefficient was found to have an important effect on required storage; positively skewed traces had the effect of reducing storage requirements by cutting short critical periods which otherwise would give rise to large storage requirements. In all cases, storage requirements were found to be extremely sensitive to the magnitude of the coefficient of variation.

The comparisons of ARMA and FFGN traces showed that for a 40 yr. life, the ARMA process closely reproduced the storage distributions derived using the FFGN generator when H was less than about 0.8. For larger values of H, the ARMA sequences gave rise to higher storage requirements, apparently because the correlation functions of the ARMA and FFGN processes begin to diverge for moderate lags at high H values.

1. INTRODUCTION

The principal purpose of this investigation was to operationally compare several different streamflow generation models that have been suggested for use in water resource system analysis. The streamflow generation models most commonly used at present are of the short memory (Markov) type; models which exhibit longer term persistence features (Hurst effect) are well understood but have not extensively been used in resource evaluation. This latter type of model is usually more expensive to operate than the simpler autoregressive models; hence it is of interest to find under what conditions of model usage one type of model is preferable to the other. The key issues to be addressed in the comparisons are sensitivity of the models to the levels of the various model parameters and to the associated parameter uncertainty.

Numerous comparative tests could be derived. One such test which incorporates the issues of flow generation, model parameters, and system demands involves use of the different types of models to determine the storage capacity of a surface water reservoir that would be needed to fully satisfy a particular water use (demand) pattern. In this work the storage distribution determined using the Thomas-Fiering "sequent peak algorithm" (Fiering, 1967) was used to compare different flow generating mechanisms. This procedure was used because it was desired to examine the physical coupling between flow generation and demand sequences. Use of synthetically generated flows in an economic comparison would not have conveniently facilitated comparison of the respective models in terms of their parameters and use.

The required storage, S, computed via the sequent peak algorithm (SPA) or any other algorithm will generally be a function of several parameters, i.e.

$$S = S(\mu_x, \sigma_x, G_x, T, STP, LTP, D, SC) \quad (1.1)$$

where

μ_x = mean flow

σ_x = flow standard deviation

G_x = flow marginal distribution skew coefficient

T = facility operating life

STP = measures of short term persistence

LTP = measures of long term persistence

D = demand pattern

SC = supply criterion

Equation 1.1 is completely general. For large demands, usually in excess of 50% of the mean annual flow, the problem is one of over-year storage. In this case a model which generates annual flow volumes suffices for the analysis. For smaller demands (and small annual variability in flows) storage is predominantly needed for within-year flow smoothing. In these instances seasonal flows, typically monthly, are needed for analysis. This latter case is not examined directly in the current report; attention is directed to the over-year storage problem.

Some justification is needed for use of the SPA for comparing the operational characteristics of different flow generation models. In any water resource system application, economic benefits result from economic demand for water; losses result from shortages or excesses (floods). On an annual time scale none of the models tested can be adequately evaluated for losses due to flooding. However, they can be readily examined for losses resulting from periods when flow is below some specific threshold level of demand. Thus, it is possible to examine the general properties of generated flows by comparing the number of"below threshold flow periods" (critical periods) for specified operating lives as well as the severity (integrated duration and deficit) for each critical period during the comparison life. The SPA computes the storage that would be needed to just supply the deficit for the most severe critical period in a particular operating time span. Thus, it provides a very convenient way to physically compare coupled flow-demand sequences over a specified operation time span where the flows can be generated by different mechanisms.

While the literature covering the problem of determining the needed capacity of a reservoir is extensive much less has been done on issues related to model comparisons. Fiering (1967) examined short-term persistence via the SPA. He examined the features of some multiple-lag flow generation models but did not specifically compare them operationally. Burges (1970) and Burges and Linsley (1971) showed that the probability distribution of storage resulting from the SPA and a large number of possible short-term persistence inflow scenarios (each scenario having the same life and facing identical demand patterns) followed the Extreme Value Type I probability distribution (Gumbel, 1958). This result is useful for comparing storage distributions resulting from different generating mechanisms given that demand patterns are identical.

Wallis and Matalas (1972) compared reservoir requirements using the SPA for flows generated by Lag-one Markov models (short-term persistence) and by Fractional Gaussian Noise (FGN). Operation lives up to 100 years were considered; the average storage for each model for various demands, Hurst coefficients and lag-one correlations were reported. They did not, however, report on differences in storages that would be encountered at typical reservoir reliability levels. Practical interest is usually focussed on facilities that would satisfy planned demand with reliabilities exceeding something in the vicinity of 90%.

O'Connell (1974) extended the Wallis & Matalas comparison by essentially repeating their experiments substituting an ARMA (1,1) model for the FGN model. His work was principally directed toward finding an economical substitute model for FGN. The results showed that the ARMA(1,1) model behaved much the same as FGN when mean reservoir storage requirements at high levels of development were compared to those derived using a Lag one Markov model.

An operational comparison using economic measures for a multiple purpose single reservoir was undertaken by Jettmar (1974). While his results showed differences in the computed economic returns, it is difficult to determine what cause and effect relationships gave rise to the differences. This work was plagued with the problem of sample estimates for generator parameters and limited to an inherently poor fit in the lag-one correlation coefficient resulting from the form of the fast fractional Gaussian Noise (FFGN) model (Mandelbrot, 1971) that was available.

In this report issues of sample estimates of model parameters were completely bypassed. In all comparisons population parameters were used to facilitate identification of model-demand couplings. Three specific models, ARMA (1,1) (Box and Jenkins, 1970), Lag-one Markov, and FFGN (Mandelbrot, 1971) were used. Work reported in the literature covering these models was extended to permit their use in the more generalized domain of three parameter log-normal marginal distributions. Annual flow model parameters examined were: mean, standard deviation, skew coefficient, lag-one correlation coefficient and the Hurst coefficient. Operation life was held fixed at 40 years. Model comparisons were made via distributions of storage needed to satisfy specified demands, the number of critical periods per trace, and the "drawdown" time for the most severe critical period per trace.

In all cases the storage distributions were determined via Monte Carlo generation. This was necessary because with long-term persistence models we anticipated that there may be insufficient critical periods per trace to satisfy the requirements of the extreme value type distribution (see Gumbel, 1958). Furthermore, at this point in time there is not a closed form solution for the theoretical storage distribution corresponding to fixed operating life, fixed demand, model type and model parameters.

It should be carefully noted that all of the models used are kinematic; they describe certain properties of observed time series. Attempts to investigate dynamic effects which cause time series to exhibit certain behavior are in their infancy. Klemes (1974) attempted to explain some of the observed long-term persistence effects in runoff records. Jackson (1975) categorizes models as prescriptive or descriptive. Klemes sought arguments to support different descriptive models. The work reported herein deals only with comparison of models which may be either prescriptive or descriptive, depending upon their intended use. This work cannot identify the "best" model for use in water resource time series modelling.

Chapter 2 covers details of the three different models. Chapter 3 examines transformations useful for extending the models to skewed marginal distribution. Chapter 4 reports on comparisons between FFGN and Lag-one Markov Models. Chapter 5 compares ARMA and FFGN models. The appendices contain model parameter transformations necessary for generating FFGN (Appendix A) and ARMA(1,1) (Appendix B) series in a log normal domain.

2. SYNTHETIC STREAMFLOW GENERATORS: THEORETICAL RELATIONSHIPS

Three synthetic flow generators were compared in this study. The simplest, and most frequently used, is the Markov model. A more sophisticated model, an approximation to the discrete fractional Gaussian noise model of Mandelbrot and Wallis (1969) was also compared. A simpler autoregressive moving average (ARMA) model which is claimed to have properties similar to fractional Gaussian noise (FGN) (O'Connell, 1974) was also compared. The mathematical forms and some properties of each of the models are given below.

2.1. Markov Models

The general Markov model has the form (following the notation of Box and Jenkins, 1970)

$$X_t = \phi_1(X_{t-1}-\mu) + \phi_2(X_{t-2}-\mu) + \dots \phi_m(X_{t-m}-\mu) + \mu + \varepsilon_t \tag{2.1}$$

where m is the order of the process, μ is the process mean, and ε_t is an independent process with mean zero and variance

$$\mathrm{Var}(\varepsilon) = (1-\rho_1\phi_1-\rho_2\phi_2 - \dots \rho_m\phi_m)\mathrm{Var}(X) \tag{2.2}$$

and where ρ_k is the process lag k correlation coefficient,

$$\rho_k = \frac{E\{(X_t-\mu)(X_{t-k}-\mu)\}}{\mathrm{Var}(X)} \tag{2.3}$$

The Yule-Walker equations (Box and Jenkins, 1970) relate the ρ's and ϕ's,

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \\ \cdot \\ \rho_m \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & & \rho_{m-1} \\ \rho_1 & 1 & \rho_1 & \rho_{m-2} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \rho_{m-1} & \rho_{m-2} & \dots & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \\ \\ \\ \phi_m \end{bmatrix} \tag{2.4}$$

The most commonly used autoregressive (Markov) model is the lag-one model, for which

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \varepsilon_t \tag{2.5}$$

$$\rho_1 = \phi_1 \tag{2.6}$$

$$\text{and } \mathrm{Var}(\varepsilon) = (1-\phi_1^2)\mathrm{Var}(X) \tag{2.7}$$

For use in generation, eq. 2.5 becomes

$$X_t - \mu = \phi(X_{t-1} - \mu) + (1-\phi_1^2)^{\frac{1}{2}} \sigma_x \eta_t \tag{2.8}$$

where σ_x^2 is the desired process variance and η_t is an independently generated random variate of mean zero and variance one from any desired probability distribution.

The autocorrelation function of the lag-one autoregressive process is given by

$$\rho_k = \rho_1^k \tag{2.9}$$

Hence, the autocorrelations decay exponentially, and, especially for annual streamflows where $\rho_1 \cong 0.2$, the correlation beyond a small number of lags is essentially zero, for instance, if $\rho_1 = 0.2$, $\rho_{10} \cong 10^{-7}$.

One final theoretical property of the autoregressive model of lag m is that all the information from the past carried forward in the present flow value is assumed to be contained in the m most recent flow values, for instance, for a Lag-one Markov model of annual flows the present year's flow is assumed to depend only on the past year's flow, and not on any of the previous flow values. There appears to be some physical basis for this assumption, since persistence in annual flows is mostly a result of carry-over effects in sub-surface storage which do not extend significantly beyond one year. Fiering (1967) gives plausible explanations for longer term carry-over effects. Klemes (1974) discusses in depth physical factors that can give rise to apparent carry-over effects at different lags.

2.2. Fractional Gaussian Noise

Prerequisite to an explanation of Fractional Gaussian Noise (FGN) is an understanding of the Hurst phenomenon. Let $\{X\}$ be the sequence of flows $X_1, X_2, \ldots X_n$. Now consider the cumulative sum of flows,

$$C_k = \sum_{i=1}^{k} X_i$$

If this cumulative sum is normalized at each time k by subtracting the average cumulative inflow over the record length, n, the normalized cumulative flow is

$$D_k = \sum_{i=1}^{k} X_i - k/n \sum_{i=1}^{n} X_i \qquad (2.10)$$

which represents, for an infinite reservoir with uniform withdrawal C_n/n the excess or deficiency relative to the initial storage level at each time k. If the maximum and minimum deviations are specified as $D_{max} = \max_k(D_k)$ and $D_{min} = \min_k(D_k)$, the range for the record length n may be defined as $R_n = D_{max} - D_{min}$. The cumulative sum and range are shown graphically in Figure 2.1.

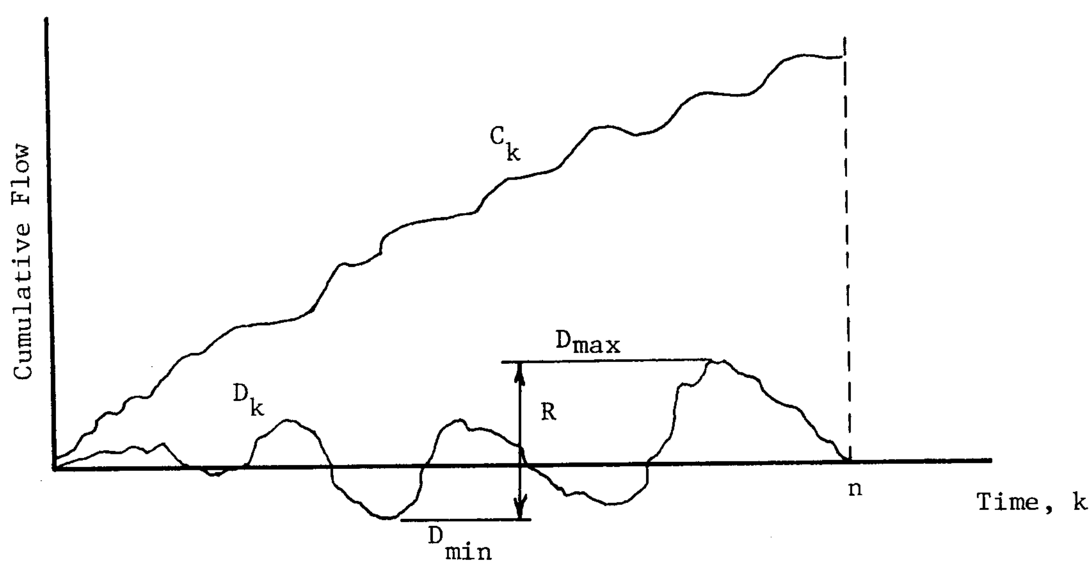


Figure 2.1. Cumulative Range of Departures

The basis for Fractional Gaussian Noise lies in an extensive study reported by Hurst (Hurst, 1951; 1956; Hurst, et.al., 1965) in which a large number of geophysical time series were investigated for the cumulative range of departures. Hurst found that a rescaled range could be defined as $R_n^* = R_n/S_n$, with S_n the standard deviation of the time series of length n, and that the rescaled range was dependent only on a single parameter H as

$$R_n^* \propto n^H \tag{2.11}$$

Using a simple coin tossing model, Hurst also showed that for a normal independent process, $E(R_n^*) = 1.25n^{.5}$, a result shown (independently) to hold without the assumption of normality by Feller (1951). The experimental results from Hurst's work showed, for about 900 annual geophysical time series, that the Hurst coefficient H had an average value of about 0.73 with a standard deviation of 0.09.

Mandelbrot and Wallis (1969) point out that the classically used Markov models, regardless of the size of the parameters and the number of lags used, lie within the Brownian domain of attraction for which assymptomatically $H = 0.5$. They have presented an alternate model, Fractional Gaussian Noise, which preserves the Hurst effect. The details of this model have been reported and discussed extensively elsewhere (Mandelbrot and Wallis, 1969; O'Connell, 1974; Klemes, 1974) and will not be repeated here, except to point out that FGN is (formally) the derivative of a smoothed Fractional Brownian motion process,

$$X_{fgn}(t,H) = \delta^{-1}(B_H(t+\delta) - B_H(t)) \tag{2.12}$$

where

$$B_H(t,\delta) = \delta^{-1}\int_t^{t+\delta} B_H(v)\,dv \tag{2.13}$$

and

$$B_H(t_2) - B_H(t_1) = (H + .5)^{-\frac{1}{2}} \{\int_{-\infty}^{t_2} (t_2-u)^{(H - .5)} dB(u)$$

$$- \int_{-\infty}^{t_1} (t_1-u)^{(H - .5)} dB(u)\} \quad (2.14)$$

The process B(u) is a Brownian motion process,

$$B(u) = \int_{-\infty}^{u} Gn(s)ds \quad (2.15)$$

with $G_n(s)$ an independent Gaussian white noise process, hence dB(u) is an independent Gaussian white noise process. Consequently, the difference between an FGN process and a Gaussian white noise process is that the white noise process may be represented as an increment of a Brownian motion process while FGN is represented as an increment of a Fractional Brownian motion process. The effect of persistence (correlation structure) on the analysis is not immediately clear, however, for a Markov process, or any process with $\sum_{j=1}^{\infty} \rho_j < \infty$, i.e., for the sum of the correlation function (or alternatively for the spectral density at zero frequency) finite, the process is essentially Brownian and has H = 0.5.

Perhaps a more straightforward explanation of FGN results when the process is considered as being defined by its autocorrelation function. For an FGN sequence of mean zero and variance one, this autocorrelation function is

$$C(s,H) = 1/2((s+1)^{2H} - 2(s)^{2H} + (s-1)^{2H}) \quad (2.16)$$

where s is the lag. The correlation function is uniquely defined by the Hurst coefficient, H, a property which has important implications discussed below. For the present, however, consider an FGN sequence with H = .7 and a lag-one Markov sequence with the same lag-one correlation. The correlation coefficients at several lags are given in Table 2.1.

Table 2.1. Comparison of Lag Correlation Coefficients for FGN and Markov Models

Lag	FGN	Markov
1	.319	.319
2	.189	.102
4	.122	.010
7	.087	3.3×10^{-4}
10	.070	1.1×10^{-5}
20	.046	1.23×10^{-10}
40	.031	1.51×10^{-20}
70	.022	2.06×10^{-35}
100	.018	2.79×10^{-50}

The FGN sequence has small (but not negligible) correlation at large lags, whereas beyond a few lags the Markov sequence has essentially zero correlation. It is this long term, or low frequency, persistence which is preserved by the Fractional noise model.

2.2.1. Discrete Approximations:

In practice, sequences of FGN must be generated on a digital computer, hence the integration limits in eq. 2.14 must be replaced by finite upper and lower cutoff points, and the integrals replaced by summations. Two early discrete approximations to FGN were proposed by Mandelbrot and Wallis (1969). These approximations were generally quite expensive computationally, and, in the case of the Type II approximation, give spurious results except when H is near one, a condition not often of practical interest. To overcome these limitations, Mandelbrot (1971) proposed a Fast Fractional Gaussian Noise (FFGN) generator. In this approach, a process is generated which has a correlation function approximately described by eq. 2.16 rather than attempting a discrete approximation to the theoretical definition of the process (eqs. 2.12-2.14). Details are provided by Mandelbrot (1971), however, briefly, FFGN consists of a high and a low frequency term, where the low frequency term is chosen to approximate the correlation structure between specified high and low frequency cutoff points, and the high frequency term is chosen to make up the deficiency in components of the correlation function above the high frequency cutoff point. The cutoff points are determined by two convenience parameters, the base B and quality factor, Q, along with the desired length of time series T. The low frequency cutoff is chosen to be beyond 1/T, the minimum frequency characteristic of the time series, while the high frequency cutoff is determined by the base B. Low frequency effects are provided by a term which is itself the weighted sum of several lag-one Markov sequences, while the high frequency deficiency is made up by a single lag-one Markov process. Although this discrete approximation is (being the sum of several Markov sequences) itself within the Brownian domain, over the finite length T of the desired time series the approximation is quite good for $H \lesssim .85$.

The low frequency term in FFGN is specified by the choice of B and Q, values of which are suggested by Mandlebrot. The values used in this study are in the recommended range, e.g., B = 3, Q = 6. The high frequency term, however, may be chosen arbitrarily to result in any desired lag-one correlation coefficient (within bounds derived below) as follows:

note that for a zero mean unit variance FFGN sequence,

$$C(s,H) = Cov_L(s,H) + \sigma_h^2 \rho_h^s \tag{2.17}$$

where ρ_h is the high frequency lag-one correlation coefficient, and σ_h^2 is the high frequency variance (uniquely determined by B, Q, H, and T). Given a desired value of the population lag-one correlation coefficient, and noting that for a process of variance one $\rho(s,H) = C(s,H)$,

$$\frac{\rho(1,H) - Cov_L(1,H)}{\sigma_h^2} = \rho_h \tag{2.18}$$

where, for $o \leq \rho_h \leq 1$

$$Cov_L(1,H) \leq \rho(1,H) \leq Cov_L(1,H) + \sigma_h^2 \tag{2.19}$$

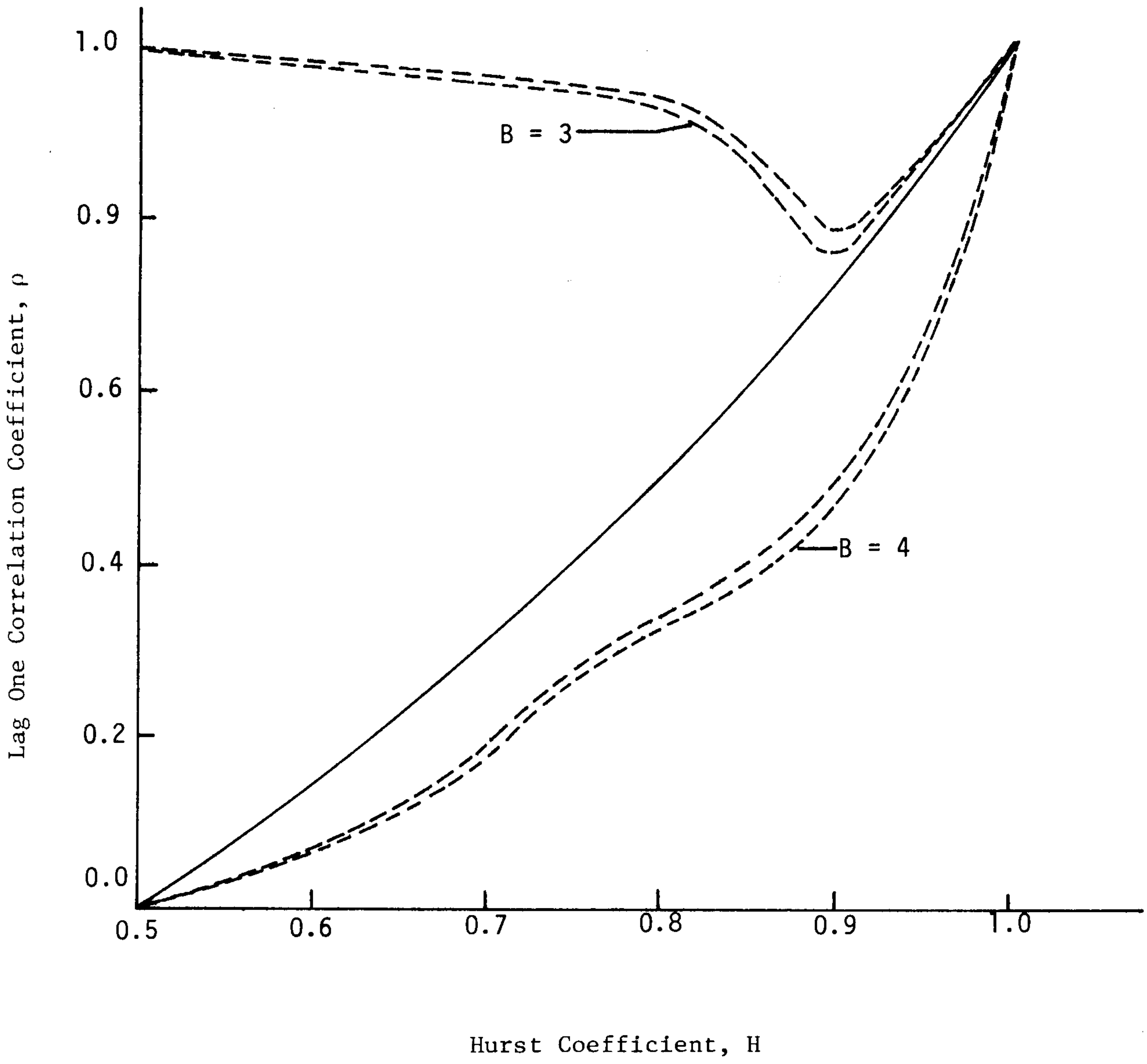


Figure 2.2. Feasible Combinations of ρ and H for FFGN, Sequence Length = 50, Q = 6

This feasible range is plotted in Figure 2.2. The two bounds plotted correspond to different values of the base, B, which is seen to have only a relatively small effect on the feasible region. The time series length, T and the quality factor, Q were found to have even less effect on the width of the feasible region. The solid line in Figure 2.2 corresponds to theoretical FGN (eq. 2.16), and is approximately the nominal lag-one correlation coefficient provided by FFGN.

It may be seen in Figure 2.2 that, assuming a high frequency term consisting of a lag-one Markov series with positive lag-one correlation, it is impossible to generate FFGN sequences with high H (e.g., $H \sim .85$) and even moderate lag-one correlation (e.g. $\rho \sim 0.2$) as often are observed in practice. It should be noted that this high frequency term has essentially no effect on preservation of the Hurst phenomenon, which depends on low frequency behavior, even though adjustment of the high frequency term results in deviation of the correlation function from that defined by eq. 2.16 for low lags. The concern is, then, purely a practical one, as lag-one correlation coefficients of greater than about 0.35 are rarely observed in annual flow series. One possible approach might be sampling from the spectrum (Mejia and Rodriguez, 1974b), however, this approach is not, at present, computationally economic. In this study we have taken only population lag-one correlation coefficients lying in the region given by Figure 2.2 for B = 3, however, the constraint on ρ should be recognized as a limitation of FFGN in operational studies.

2.3. Autoregressive Moving Average Models

O'Connell (1971;1974) has proposed the use of autoregressive moving average models to synthesize stochastic streamflow sequences. O'Connell shows that the Hurst effect may be preserved for finite length time series using ARMA models, and claims several advantages for the ARMA models over FFGN, principally computational savings.

The general ARMA model of order (p,q) is defined as (for a zero mean process)

$$(1 - \phi_1 B - \phi_2 B^2 - \dots \phi_p B^p) X_t = (1 - \theta_1 B - \theta_2 B^2 - \dots \theta_q B^q)\varepsilon_t \qquad (2.20)$$

where

$$B^k X_t = X_{t-k}$$

and ε_t is a white noise process of mean zero.

In O'Connell's work ARMA (1,1) processes were found to give results about as good as the higher order ARMA processes he tested. In this work only ARMA (1,1) processes are investigated. The ARMA (1,1) model is

$$(1-\phi_1 B)X_t = (1 - \theta_1 B)\varepsilon_t \qquad (2.21)$$

Rewriting the process in terms of the noise component, and taking $\phi = \phi_1$, $\Theta = \Theta_1$

$$X_t = (1 - \theta B)(1 - \phi B)^{-1}\varepsilon_t$$

$$= (1 + (\phi - \theta)B + \phi(\phi - \theta)B^2 + \dots \phi^{k-1}(\phi - \theta)B^k + \dots)\varepsilon_t \qquad (2.22)$$

Hence, noting that $Var(B^k \varepsilon_t) = Var(\varepsilon_t)$ by virtue of the independence of ε_t,

$$Var(X) = 1 + (\phi - \theta)^2 \sum_{k=1}^{\infty} \phi^{2k}\, Var(\varepsilon) = (1 - \phi^2)^{-1}(1 - 2\phi\theta + \theta^2)Var(\varepsilon) \qquad (2.23)$$

For generating synthetic flows, eq. 2.21 is rewritten

$$X_t = \mu + \phi\,(X_{t-1} - \mu) + \varepsilon_t - \Theta\,\varepsilon_{t-1} \qquad (2.24)$$

where ε is an independent process with mean zero and variance

$$Var(\varepsilon) = (1 - \phi^2)(1 - 2\phi\theta + \theta^2)^{-1} Var(X) \qquad (2.25)$$

where $Var(X)$ is the desired process variance and μ is the desired process mean.

The correlation structure of the ARMA (1,1) model may be easily derived from eq. 2.21, and is

$$\rho_1 = (1 - \phi\theta)(\phi - \theta)(1 + \theta^2 - 2\phi\theta)^{-1}$$

$$\rho_j = \phi^{j-1}\rho_1 \qquad (j \geq 2) \qquad (2.26)$$

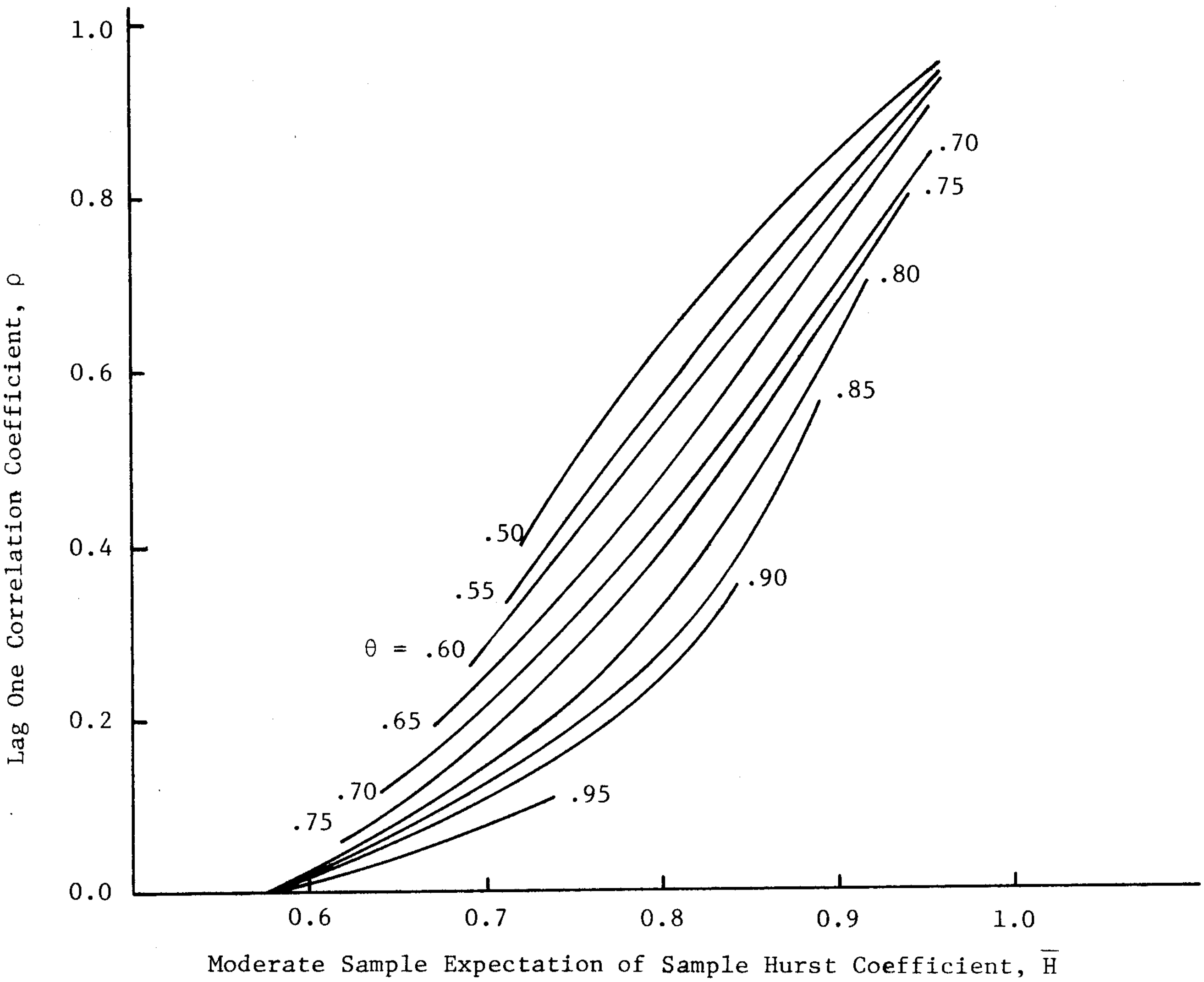
This correlation structure allows the first two correlation coefficients to be "fitted", rather than only the first as in the Lag one Markov model. Nevertheless, the model is clearly within the Brownian domain of attraction as

$$\sum_{j=1}^{\infty} \rho_j = \rho_1(1 + \sum_{j=1}^{\infty}\phi^j)$$
$$= \rho_1(1 + (1 - \phi)^{-1}) \qquad (2.27)$$

which is clearly finite so long as $\phi \neq 1$. However, for a finite length time series it is possible to select ϕ very nearly equal to one to retain low frequency persistence, while choosing θ to match high frequency behavior. An extensive analysis of the use of ARMA (1,1) models to model low frequency characteristics of streamflows is given in O'Connell (1974). The principle difficulty in using the ARMA (1,1) model is that the Hurst coefficient, H, is not included explicitly but must be estimated from Monte Carlo experiments. Unfortunately, both the frequently used estimates of H are biased (Wallis and Matalas, 1970), even for relatively large n. Consequently, there is no population value of H for the ARMA (1,1) process. O'Connell found that the estimated values of H from ARMA (1,1) processes tended first to rise with increasing sequence length (for $H \gtrsim .7$) then to fall, the rise corresponding to a reducting in bias and the drop due to a return to the influence of the basic Brownian nature of the ARMA (1,1) process. Estimated long sequence length values of $\bar{H}$ corresponding roughly to the maximum value of H with n for ϕ and θ fixed are given by O'Connell and are used here as pseudo-population statistics.

Using the values of $\bar{H}$ given by O'Connell, the feasible region of $\bar{H}$ and ρ is plotted in Figure 2.3. Figure 2.3 is essentially a graphical display of the information given in Table 3.1 of O'Connell (1974) which is reproduced below as Table 2.2 for convenience. A family of curves results because several combinations of ϕ and θ will yield the same $\bar{H}$. Figure 2.3 must be used with Table 2.2 as each curve is parameterized by ϕ, given in Table 2.2.

Figure 2.3 shows that the feasible region for the ARMA (1,1) model includes lower values of ρ for moderate to high values of $\bar{H}$ than does the similar



Note: Corresponding values of ϕ are given in Table 2.2.

Figure 2.3. Values of ρ and $\bar{H}$ for Specified θ for ARMA(1,1) Models; Data From O'Connell(1974).

Table 2.2. Values of $\bar{H}$ and ρ for θ and ϕ[a].

ϕ		.99	.98	.97	.95	.90	.85	.80
θ = .95	ρ_1	.111	.051	.026	.000	-.038	-.076	-.094
	$\bar{H}$	.737	.676	.634	.576	.490	.439	.403
θ = .90	ρ_1	.350	.205	.139	.073	.000	-.042	-.076
	$\bar{H}$	.839	.781	.737	.673	.576	.515	.472
θ = .85	ρ_1	.562	.384	.287	.179	.061	.000	-.044
	$\bar{H}$	.892	.840	.800	.738	.639	.576	.529
θ = .80	ρ_1	.706	.540	.433	.300	.140	.057	.000
	$\bar{H}$	.920	.875	.839	.782	.687	.623	.576
θ = .75	ρ_1	.797	.659	.558	.418	.229	.126	.055
	$\bar{H}$	.937	.897	.864	.812	.723	.661	.614
θ = .70	ρ_1	.856	.745	.657	.523	.322	.203	.119
	$\bar{H}$	.947	.911	.881	.833	.750	.690	.645
θ = .65	ρ_1	.895	.807	.732	.612	.411	.282	.188
	$\bar{H}$	.953	.920	.892	.848	.770	.714	.670
θ = .60	ρ_1	.921	.851	.789	.684	.493	.360	.260
	$\bar{H}$	.956	.926	.900	.858	.785	.732	.691
θ = .55	ρ_1	.939	.883	.832	.742	.566	.435	.331
	$\bar{H}$	.959	.929	.904	.866	.797	.747	.707
θ = .50	ρ_1	.952	.907	.864	.788	.629	.503	.400
	$\bar{H}$	.960	.932	.908	.871	.806	.759	.721

Each value of $\bar{H}$ is the mean of ten values of H, each derived as the slope of a least squares line fitted to the mean values of R_n/S_n for values of $n_s \leq 1000$ given in figure (3.12). The total sample size in all cases was 9000.

[a]Reproduced directly from Table 3.1 in O'Connell (1974)

plot (Figure 2.2) for FFGN. Unfortunately, it is not known what influences the bias in $\overline{H}$ has on these estimates. A more detailed assessment of this problem must await the Monte Carlo results of Chapter 5.

3. THREE PARAMETER LOG NORMAL TRANSFORMATIONS

3.1 Introduction

The three parameter log normal probability distribution was used throughout this report to generate sequences with marginal distributions having non-zero skew. The basic form of the distribution is

$X_t = a + \exp(Y_t)$ with $\underline{a}$ the so-called "third parameter"

where[a]

$$Y_t \sim N(\cdot|\mu_y, \sigma_y)$$

and

$$\mu_y = f_1(\mu_x, \sigma_x, G_x)$$

$$\sigma_y = f_2(\mu_x, \sigma_x, G_x)$$

$$a = f_3(\mu_x, \sigma_x, G_x) \qquad (3.1)$$

with μ_x, σ_x, and G_x the mean, standard deviation, and skew of the desired marginal distribution. The functional forms of f_1, f_2, and f_3 are given in Burges, et.al. (1975) and are not repeated here.

The principal advantages of the three parameter log normal distribution are that it can preserve the first three central moments (mean, standard deviation, and skew coefficient) of any distribution and that it is generated from a sequence {Y} with a normal marginal distribution. This second property is very helpful in deriving sequences with skewed marginal distributions. The primary difficulty with the three parameter log normal distribution is that some combinations of μ_x, σ_x, and G_x (namely those that result in $a \gtrsim \mu_x$) give substantial fractions of negative values, which is undesirable in synthetic flow generation. Bates, et. al. (1974) give the percentage of negative values generated for each parameter combination. This information was used to aid in selecting physically reasonable test cases.

---

[a]The notation $N(\cdot|\mu,\sigma)$ denotes a normal distribution with mean μ and standard deviation σ.

3.2 Lag-one Markov Models

Mejia and Rodriguez (1974a) show that an earlier result for the transformation of the lag-one correlation coefficient of a lag-one Markov model through a three parameter log normal transformation (Matalas, 1967) holds for any correlation function at any lag, τ i.e.,

$$\rho_y(\tau) = \ln(1 + \rho_x(\tau)\ (\exp(\sigma_y^2) - 1))\ \sigma_y^{-2} \tag{3.2}$$

The form of the transformation is particularly simple as the only transform parameter is σ_y, the standard deviation of the process in the log transform (normal) domain. In practice for parameter combinations of interest σ_y lies in the range $0 \leq \sigma_y \leq 1$.

If {Y} is modelled as a lag-one Markov sequence with $\rho_y = \rho_y(1)$ chosen to yield a desired ρ_x, the correlation function of {X} will not be Markov but will be distorted as

$$\rho_x(k) = \frac{\exp(\rho_y^k \sigma_y^2) - 1}{\exp(\sigma_y^2) - 1} \tag{3.3}$$

whereas if {X} were Markov the correlation function would be

$$\rho_x'(k) = \left[\frac{\exp(\rho_y \sigma_y)^2 - 1}{\exp(\sigma_y^2) - 1}\right]^k = [\rho_x(1)]^k \tag{3.4}$$

Mejia and Rodriguez then go on to show that if {Y} is generated as an ARMA (1,1) process with

$$\phi_1 = \rho_x$$

and θ_1 the solution to

$$\frac{\exp(\sigma_y^2) - 1}{\sigma_y^2} = \frac{(1 - \phi_1\theta_1)(\phi_1 - \theta_1)}{(1 + \theta_1^2 - 2\phi_1\theta_1)\phi_1}, \tag{3.5}$$

the difference between the generated correlation function (eq. 3.3) and the ideal lag-one Markov correlation structure may be reduced substantially.

Unfortunately, the illustrative example used in their work is in error and implies a much more substantial change in the correlation function than is actually the case. The results of the example used are presented below along with the corrected results as Table 3.1. Table 3.2 shows a similar computation when $\sigma_y = 0.6$.

Table 3.1. Comparison of Methods for Generating Log Normal Lag-One Markov Sequences for $\rho_x = 0.2$, $\sigma_y = 0.3$.

Lag	Theoretical Lag-One Markov	Transformed Markov[a]	Transformed Markov[b]	Transformed ARMA (1,1)
1	2×10^{-1}	2×10^{-1}	2×10^{-1}	2.02×10^{-1}
2	4×10^{-2}	4.12×10^{-2}	4.12×10^{-2}	4.01×10^{-2}
3	8×10^{-3}	8.52×10^{-3}	8.52×10^{-3}	8.0×10^{-3}
4	1.6×10^{-3}	1.77×10^{-3}	1.77×10^{-3}	1.6×10^{-3}
5	3.2×10^{-3}	3.66×10^{-3}	3.66×10^{-4}	3.2×10^{-4}
10	1.02×10^{-7}	1.40×10^{-3}	1.40×10^{-7}	1.02×10^{-7}
15	3.28×10^{-11}	5.31×10^{-3}	5.37×10^{-11}	3.28×10^{-11}
50	1.12×10^{-35}	6.50×10^{-3}	6.51×10^{-35}	1.12×10^{-35}
100	1.27×10^{-70}	4.64×10^{-3}	4.43×10^{-69}	1.27×10^{-70}

[a]Results from Mejia and Rodriguez (1974a)

[b]Corrected results for transform of lag-one Markov model

Table 3.2. Comparison of Methods for Generating Log Normal Lag-One Markov Sequences for $\rho_x = 0.2$, $\sigma_y = 0.6$

Lag	Theoretical Lag-One Markov	Transformed Markov	Transformed ARMA (1,1)
1	2×10^{-1}	2×10^{-1}	2.09×10^{-1}
2	4×10^{-2}	4.47×10^{-2}	4.03×10^{-2}
3	8×10^{-3}	1.02×10^{-2}	8.01×10^{-3}
4	1.6×10^{-3}	2.36×10^{-3}	1.6×10^{-3}
5	3.2×10^{-3}	5.45×10^{-4}	3.2×10^{-4}
10	1.02×10^{-7}	3.57×10^{-7}	1.02×10^{-7}
15	3.28×10^{-11}	2.34×10^{-10}	3.28×10^{-11}
50	1.12×10^{-35}	1.22×10^{-32}	1.12×10^{-35}
100	1.27×10^{-70}	1.80×10^{-64}	1.27×10^{-70}

The ARMA approximation gives an obviously superior fit at large lags with a small sacrifice in the lag-one correlation coefficient. However, the distortion resulting from the transformation of a lag-one Markov model is many orders of magnitude less than the difference between the ideal Markov model and an FGN model at large lags (Table 2.1) and may well be insignificant. In this work we have chosen to preserve (exactly) the lag-one correlation coefficient and so have generated lag-one Markov sequences in the Y (normal) domain with ρ_y derived from eq. 3.2 with $\tau = 1$.

3.3. Three Parameter Log Normal Transformation of Fast Fractional Gaussian Noise

The FFGN generator used in this work yields normal sequences, hence to generate synthetic sequences with skewed marginal distributions the FFGN sequences must be transformed. It is desirable to preserve the value of the Hurst coefficient and the lag-one correlation coefficient through the transformation. The lag-one correlation coefficient is easily preserved by generating sequences with $\rho_y(1)$ given by eq. 3.2. So long as this correlation coefficient is in the range given by Figure 2.2, the process is feasible. Sequences generated in the logarithmic domain have a slightly higher correlation at any given lag that do the corresponding transformed sequences, so a quick verification of feasibility can be provided by checking the desired lag-one correlation coefficient in the X (real) domain against the desired H_x using Figure 2.2.

To determine the necessary value of H in the Y domain necessary to generate a sequence {X} with specified H_x, it is necessary to investigate the distortion resulting in the correlation function of an FGN sequence caused by the transformation. While it will not be possible to exactly preserve the correlation structure given by eq. 2.16, it is possible to choose H_y so that the derived (skewed) sequence has the same correlation coefficient as does theoretical FGN for any (one) specified lag, k. The value of H_y necessary to achieve this fit is determined as

$$C(k, H_x) = T(C(k,H_y), \sigma_y)$$

where

$$C(k,H) = 1/2(\ (k+1)^{2H} - 2k^{2H} + (k-1)^{2H})$$

and

$$T(\rho,\sigma_y) = \frac{\exp(\sigma_y^2\rho) - 1}{\exp(\sigma_y^2) - 1} \qquad (3.6)$$

with H_x = desired effective Hurst coefficient of skewed data

H_y = necessary Hurst coefficient in log (normal) domain

σ_y = standard deviation of Y (normal) sequence

Equation 3.6 may be solved using Newton's method. Results are given in tabular form in Appendix A for several values of k and σ_y.

Tables 3.3a and 3.3b list the errors of approximation of FGN by FFGN (E_1) and the error in transformation of FFGN using a three parameter log normal transformation (E_2). Both errors are multiplied by 10,000 for ease of comparison.

Table 3.3a. Comparison of Transformation and Generation Errors for[a] $\sigma_y = 0.3$

H_x =	.55		.65		.75		.85	
Lag	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
0	0	0	0	0	0	0	0	0
1	0	0	0	0	13	0	208	0
2	63	-12.6	141	0	153	7.0	291	53
3	20	-6.1	57	4.8	80	14.0	256	57
4	4	-1.8	13	5.5	29	16.0	222	54
7	-3	0.4	-16	3.5	-2	12.0	194	44
10	-2	0.3	-8	2.2	-1	9.0	195	38
40	0	0.0	-2	-0.1	5	1.5	199	19
70	0	0.0	-1	-0.3	7	0.3	200	13.6
100	0	0.0	-1	-0.3	8	-0.2	201	10.7

Table 3.3b. Comparison of Transformation and Generation Errors for[a] $\sigma_y = 0.6$

H_x =	.55		.65		.75		.85	
Lag	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
0	0	0	0	0	0	0	0	0
1	0	0	0	0	13	0	208	0
2	63	-51	141	-1	153	27	291	204
3	20	-26.8	57	17.1	80	57	256	233
4	4	-9	13	20.8	29	64	222	224
7	-3	1.0	-10	13.5	-2	50	194	186
10	-2	0.7	-8	8.3	-1	38	195	160
40	0	-0.1	-2	-0.6	5	7.5	199	81
70	0	-0.1	-1	-1.2	7	2.4	200	59
100	0	-0.1	-1	-1.3	8	0.5	201	46.7

[a] E_1 is defined as the difference between Theoretical FGN and Untransformed FFGN, E_2 as the difference between untransformed and transformed FFGN, hence, $E_1 + E_2$ gives the total error of approximation.

The values of E_1 were taken from Mandelbrot (1971) while E_2 was derived numerically using values of H_y for k = 20 given in Appendix A. It should be noted that the values of H_y given in Appendix A were derived for a transformation of theoretical FGN, while the values of E_2 given in Tables 3.3 are errors resulting from transformation of FFGN, hence the values of E_2 are not zero at lag 20.

Tht total error of approximation of an FGN sequence with three parameter log normal marginal distribution is given by the sum of E_1 and E_2. Except at large values of H and σ_y, the transformation error is generally less than the error of approximation of FFGN, and in some cases the errors cancel to some extent. For the largest H value (H = 0.85) the FFGN approximation error is quite large as is the transformation error (especially for $\sigma_y = 0.6$); for sequences with higher H and σ_y the approximation deteriorates rapidly. For the more moderate parameter ranges often of interest the method suggested here should be quite adequate, and is used in the simulation studies reported in Chapter 4.

3.4. Three Parameter Log Normal Transformations of ARMA (1,1) Models

The transformation of ϕ_1 and θ_1[a] in the ARMA (1,1) process is achieved by preserving the lag-one and lag-two correlation coefficients in the transformation. For an ARMA (1,1) process

$$\rho(1) = \frac{(1-\phi\theta)(\phi-\theta)}{1 + \theta^2 - 2\phi\theta} \tag{3.7}$$

$$\rho(2) = \phi\rho(1) \tag{3.8}$$

If an ARMA (1,1) process is generated in the Y domain and transformed, preservation of the first two correlation coefficients for specified ϕ_x and θ_x leads to

---

[a] ϕ_1 and θ_1 are referred to throughout this section as ϕ and θ

$$\rho_x(1) = \frac{\exp(\rho_y(1)\sigma_y^2) - 1}{\exp(\sigma_y^2) - 1} \tag{3.9}$$

$$\rho_x(2) = \frac{\exp(\phi_y\rho_y(1)\sigma_y^2) - 1}{\exp(\sigma_y^2) - 1} \tag{3.10}$$

$$\rho_y(1) = \frac{\ln(1 + \rho_x(1)(\exp(\sigma_y^2) - 1))}{\sigma_y^2} \tag{3.11}$$

$$\phi_y\rho_y(1) = \frac{\ln(1 + \rho_x(2)(\exp(\sigma_y^2) - 1))}{\sigma_y^2} \tag{3.12}$$

hence $$\phi_y = \frac{\ln(1 + \rho_x(2)(\exp(\sigma_y^2) - 1))}{\ln(1 + \rho_x(1)(\exp(\sigma_y^2) - 1))} \tag{3.13}$$

letting $\rho_y(1) = C = C(\phi_x, \theta_x, \sigma_y)$ (all known values) then,

$$\frac{(1 - \phi_y\theta_y)(\phi_y - \theta_y)}{1 + \theta_y^2 - 2\phi_y\theta_y} = C$$

which may be written $$\theta_y^2 + A\theta_y + 1 = 0 \tag{3.14}$$

with $$A = \frac{\phi_y^2 + 1 - 2\phi_y C}{C - \phi_y}$$

Of the two resulting roots, only one satisfies the stationarity condition (Box and Jenkins, 1970)

$$|\theta_y| \leq 1.$$

Appendix B gives tables of ϕ_y and θ_y for given values of ϕ_x, θ_x, and σ_y.

A further constraint is imposed on the solution of eq. 3.14 by the requirement that $A^2 \geq 4$. Although a solution may always be found for ϕ_y from eq. 3.13, some combinations of ϕ_x and θ_x result in imaginary solutions for θ_y. Figure 3.1 shows the upper bound of the feasible range for θ_x as a function of ϕ_x and σ_y. This constraint is important only for $\phi_x \leq \theta_x$ which results in $\rho_x(1) \leq 0$, so will not often be important in practice.

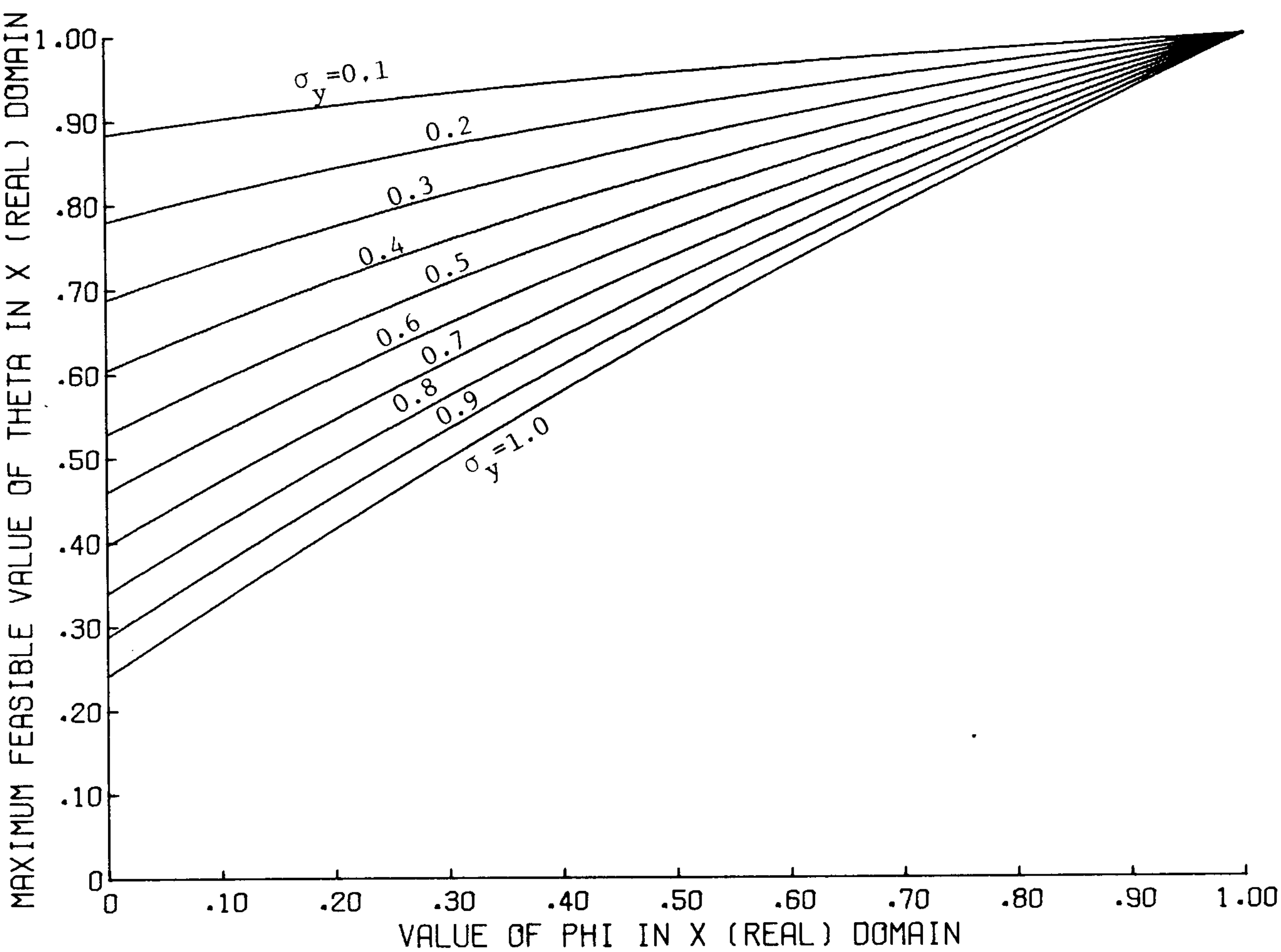


Figure 3.1. Upper Limit of Feasible Parameter Range For Approximating a Three Parameter Log Normally Distributed ARMA(1,1) Sequence.

The distortion in the autocorrelation function resulting from transformation of three parameter log normal ARMA (1,1) sequences is given in Table 3.4. The results were obtained using the values of ϕ_y and θ_y given in Appendix B, calculating the correlation coefficients using eq. 2.16, and transforming using eq. 3.3. The errors were calculated as the difference between the theoretical ARMA (1,1) correlations calculated from eq. 2.26, and the transform results. If the transform parameters given in Appendix B were exact, the errors for lags one and two would be zero, however, some discrepancy results from truncation error. Nevertheless, the distortion caused by the transformation is less than that in the transformation of FFGN sequences (Table 3.3), and may be considered negligible for the purposes of this work.

Table 3.4. 10,000 Times the Error of Approximation in the Autocorrelation Function of a Three Parameter Log Normally Distributed ARMA (1,1) Sequence Obtained Through Transformation of a Normally Distributed ARMA (1,1) Sequence.

	$\sigma_y = 0.3$			$\sigma_y = 0.6$		
sequence[a]	I	II	III	I	II	III
Lag						
1	-1.0	0.1	-1.9	0.4	1.3	0.7
2	-0.9	0.1	-1.9	0.4	1.3	0.7
3	-0.9	0.1	-1.9	-0.2	1.1	0.7
4	-1.0	0	-1.9	-0.9	0.8	0.7
7	-1.3	-0.3	-1.9	-3.1	-0.4	0.7
10	-1.4	-0.7	-1.9	-4.2	-2.1	0.7
40	-0.1	-2.2	-2.0	-0.3	-9.1	-0.1
70	0	-1.1	-1.9	0	-4.7	-0.8
100	0	-0.4	-1.6	0	-1.7	-1.2

[a]Sequence
I $\phi_x = 0.85$ $\theta_x = 0.75$
II $\phi_x = 0.95$ $\theta_x = 0.85$
III $\phi_x = 0.985$ $\theta_x = 0.95$

4. OPERATIONAL COMPARISONS: FFGN AND MARKOV MODELS

4.1 Introduction

Results of operational comparisons of FFGN and Markov generated flow sequences are presented in this chapter. The comparison is extended to include ARMA (1,1) models in Chapter 5. The principal comparative measure of the flow generators was the estimated cumulative probability distribution of storage that would be required to satisfy a specified demand over a given operating period. Individual cumulative distributions were determined as follows: A large number of synthetically generated flow traces (usually 1,000), with each trace of length equal to the operating life of an hypothetical reservoir, was generated. Each trace (e.g., the m'th trace) was then processed through the Thomas-Fiering (Fiering, 1967) sequent peak algorithm (SPA) to yield a value of required storage, S_m, such that if the reservoir were full at the onset of the worst "critical period" of the trace then the stored water plus inflow would be just sufficient to fully satisfy demand. The ordered S_m for all traces (using an appropriate plotting rule) then form the empirical probability distribution of interest. In this work the plotting rule used was that the i'th largest storage, S_i, was plotted at cumulative probability level $1 - \frac{i}{N+1}$

The SPA might be criticized because it requires the assumption that the initial contents of the reservoir (whose size is unknown before a complete pass through the algorithm is made) be greater than or equal to zero. In practice this requirement is not too restrictive because most facilities when brought on line have some water stored but are not necessarily full.

Because it seemed likely that traces generated by FGN models would exhibit longer periods of above average and below average flow (i.e., greater long-term persistence) than would be found in traces generated by short memory Markov models it was decided to examine additional features of the generated traces. The principal concern with use of the SPA to develop an index for comparing traces was that it concentrates on the most severe critical period per trace rather than the longest critical periods. It was decided to examine how often the most severe critical period coincided with the longest critical period per trace. The number of critical periods per trace was also examined. By making use of intermediate computations in the SPA we obtained, for each trace, the number of critical periods (i.e., the number of "troughs" per trace), duration of the most severe critical period, time to fill following the most severe critical period, duration of the longest critical period, and time to fill following the longest critical period. These quantities were examined to find differences attributable to flow generation parameters and models.

Of particular interest is the form of termination of periods of sub-demand levels of flow. It was not clear at the beginning of the investigation if long-term persistence implied that draw-down and fill-up times would be approximately the same or alternatively if fill-up could be significantly influenced by "Noah" (extreme) type events. This issue is discussed in Section 4.3 of this chapter.

4.2 Use of Storage Distribution Diagrams

Burges and Linsley (1971) showed that the extreme value Type I probability distribution (Gumbel, 1958) described the storage distribution resulting from Markov generation and use of the SPA algorithm. In the current work it was found that the extreme value distribution, in general, suitably described the empirically plotted storage distributions for most of the cases examined. Where the theoretical Gumbel distribution was inappropriate the best "by eye" curve was determined. This could usually be effectively done via two straight lines drawn on extreme value paper. All storage distributions (Figures 4.1-4.13) are drawn on extreme value paper such that the theoretical curves corresponding to the empirical cumulative probability distributions plot as straight lines. This display is useful for comparing

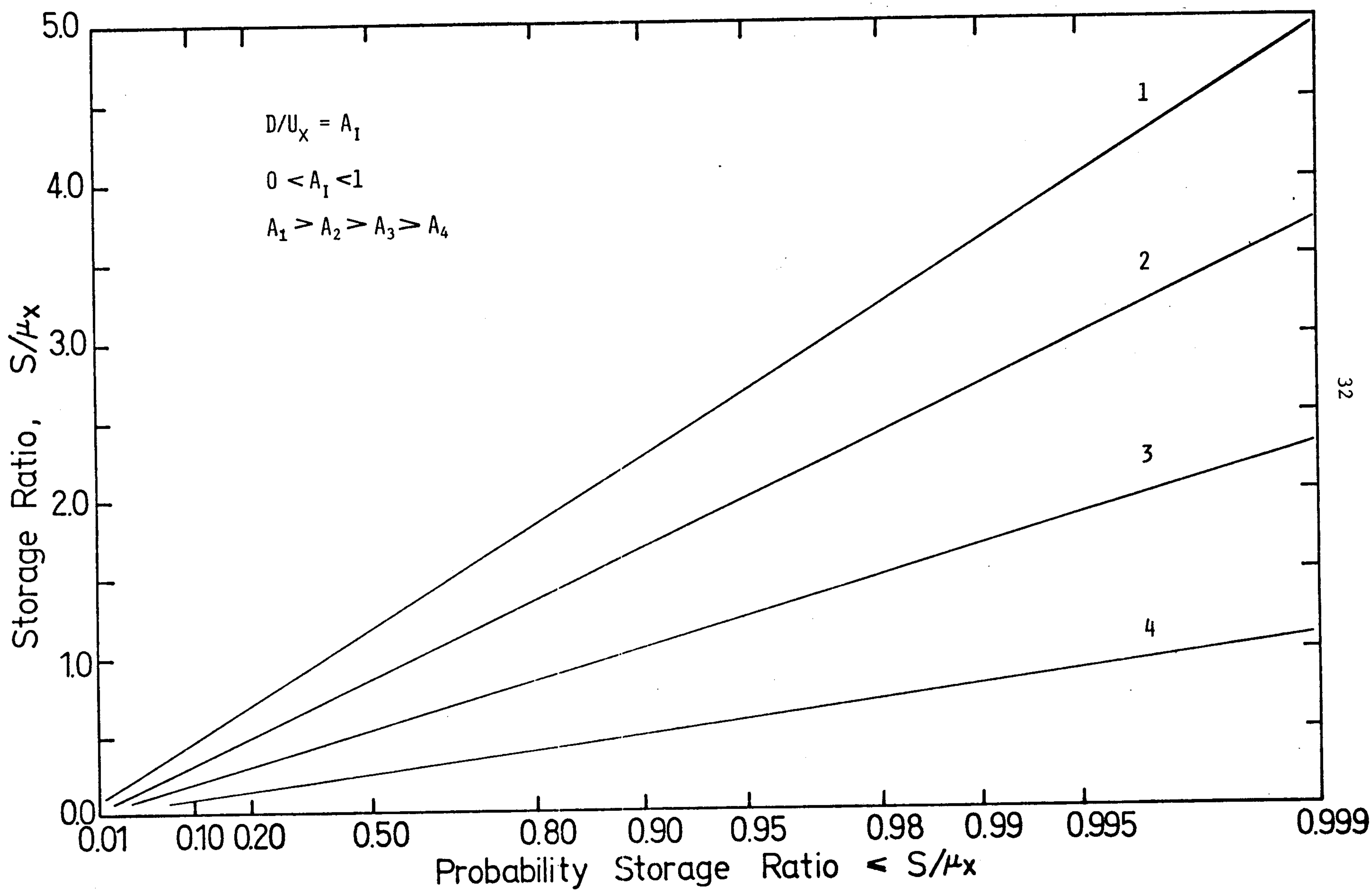


Figure 4.1. Hypothetical Storage Distributions Resulting From a Particular Flow Generator, Specific Operating Life and Four Demand Patterns.

the coupled generation-demand effects of the different models and their parameters.

The probability-storage diagram is useful in any particular situation for determining what levels of flow development are realistic and, for a given demand level, estimating the reservoirs' reliability in meeting the demand schedule. Four such curves are given in Figure 4.1. In this figure (and in all subsequent probability-storage plots) demand is expressed nondimensionally as annual demand divided by the mean annual flow, i.e. $D^* = D/\mu_x$. Demand was held constant over the operating life of the hypothetical single reservoirs used in this work.

In Figure 4.1 each curve results from a different demand level. Curve 1 ($D^* = A_1$) results from the largest of the demand sequences. If, for example, it were practical to construct a facility whose capacity was equal to the mean annual flow of the river (i.e., $S^* = S/\mu_x = 1.0$) then demand $D^* = A_1$ could be supplied with a reliability of about 40% while the much smaller demand $D^* = A_4$ (Curve 4) could be supplied with a reliability of about 0.997. Here 0.997 reliability means that of all possible inflow scenarios (based on the assumed or fitted flow generation model) 99.7% of the inflow traces, each trace of length equal to the operating life, would have sufficient flow plus buffer storage to fully satisfy demand. The remaining 0.3% of all cases would not fully satisfy demand and some supply shortage would result. Another way of interpreting the curves of Figure 4.1 is to consider the development of a water supply whose desired reliability for the proposed operating life is, for example, 99%. To achieve this level, demand A_4 would require a storage $S^* = 0.82$ while demand A_1 would require storage $S^* = 3.63$. This latter quantity would usually be unrealistic, hence meeting demand schedule A_1 at reliability level 0.99 would be an impractical objective.

Section 4.3 presents storage-probability relationships obtained from the numerical experiments that were made to compare different flow generation models and model parameters. Figures 4.2-4.13 contain representative information from all of the Markov and FFGN sequences examined in this work.

4.3 Numerical Results

In all cases reported herein the operating and economic lives were both taken as 40 years. This figure was adopted because economic returns over longer periods usually have negligible impact on project evaluation. Additionally, choice of a 40 year operating period is representative of practical aspects of planning. Several test cases were run for lives of 100 years with results essentially the same as those for 40 year lives presented herein (inputs were generated from identical models) except for scale. It should be noted that the increase in scale is such as to make reservoir storage requirements unrealistically large at all but very low demands for lives of 100 years.

Each curve in Figures 4.2-4.13 is based on a theoretical fit to the cumulative distribution defined by 1000 values of storage, S_m. Although there is no particular mystique associated with the number 1000, Burges and Linsley (1971) found for Markov generation that 1000 storage values defined very well storage distributions for demands up to $D^* = 0.9$ and annual flows having coefficients of variation up to 0.5. Results of the current research verified this finding with the exception that the theoretical Gumbel distribution did not provide a good fit to the data for values of Hurst coefficient above 0.8, particularly when the coefficient of variation was reduced to the neighborhood of 0.25. Three demand levels (consistent with the over-year storage problem) of $D^* = 0.5$, 0.7, and 0.9, applied uniformly over the operating life, were examined. In all cases population parameter values were used to generate the synthetic annual flows.

4.3.1 Empirical and Theoretical Storage Distributions

Figure 4.2 shows a typical result for a good fit of the extreme value

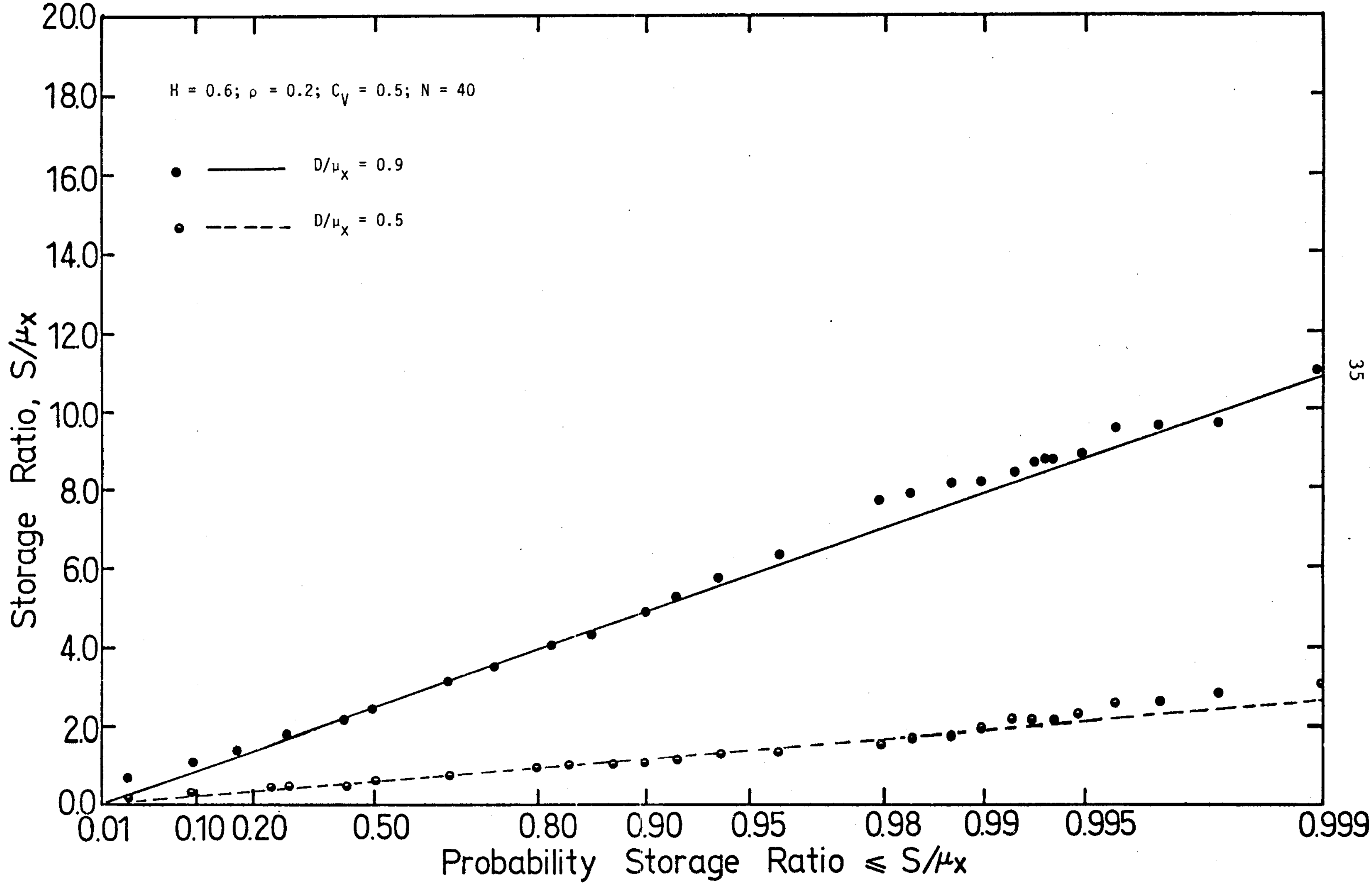


Figure 4.2. Empirical and Theoretical Extreme Value Type I Storage Probability Distributions.

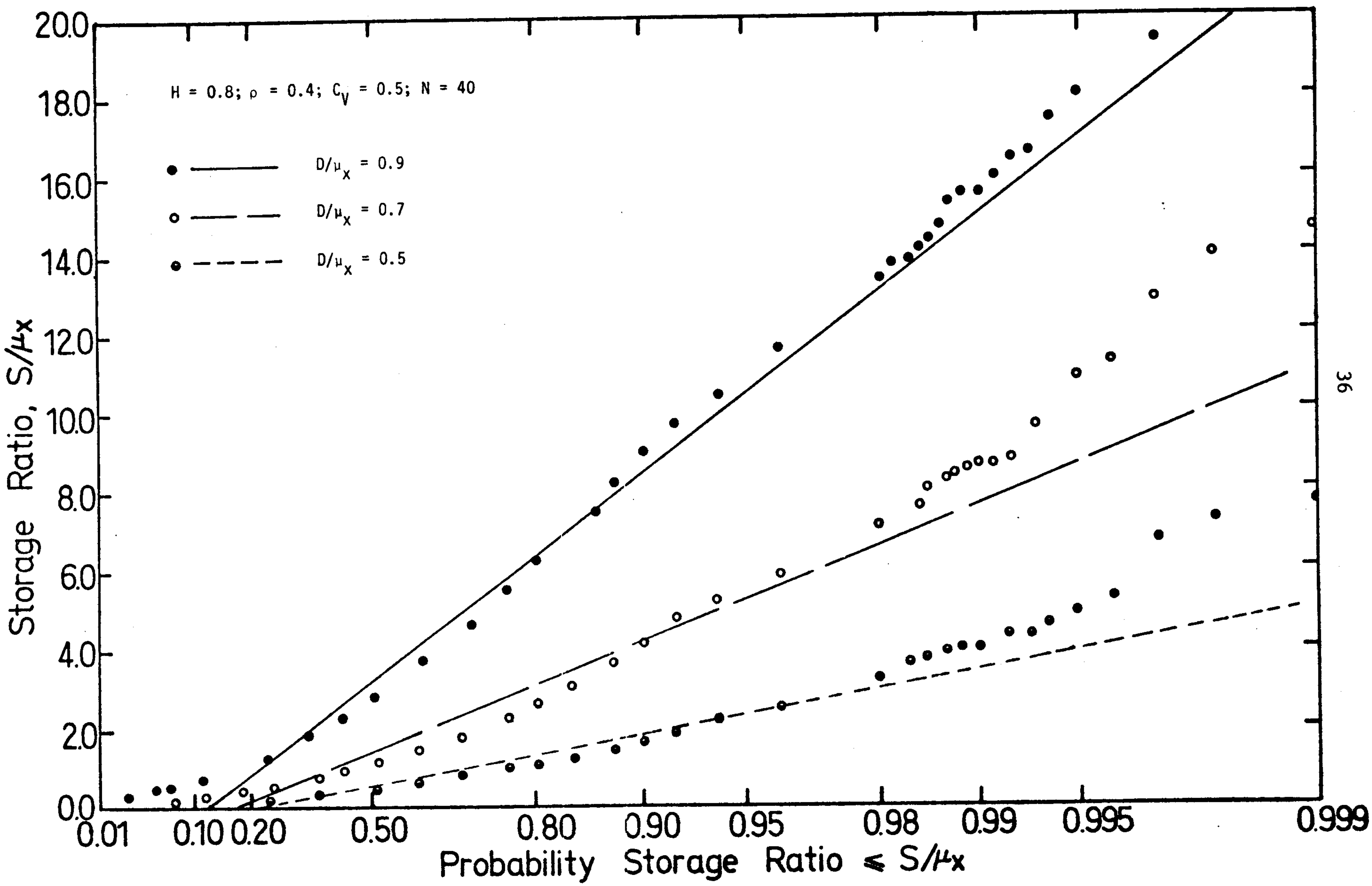


Figure 4.3. Empirical and Theoretical Extreme Value Type I Storage Probability Distributions Showing Regions of Disagreement.

theory to experimentally generated data. For obvious graphical reasons not all 1000 data points are shown in the figure; the 20 largest storage values are, however, plotted.

Figure 4.3 is representative of a poor fit of the theoretical distribution to empirical data. It appears that for this case the data are close to some threshold which might be represented by two extreme value populations for each demand, particularly at the higher demand levels. The extreme value theory was expected to be unsatisfactory for large H and relatively short life (40 years) because the long runs of above average and below average flows do not give rise to many critical periods per trace. This effect necessarily invalidates one of the foundations of the extreme value type 1 theory (Gumbel, 1958). It was found, however, that the skewness of the marginal flow distribution was quite important in breaking up these long runs. Termination of critical periods is discussed in Section 4.3.

In Figures 4.4-4.12 the straight lines are theoretical descriptions of experimentally obtained storage values. These storage values were visually compared with the theory. When a poor fit resulted approximate lines of best fit to the experimental data were determined. For example, curves F in Figures 4.10 and 4.11 are by-eye fits to the experimental data. Summary information, sufficient for constructing most of the storage distributions that were tested, are given in Table 4.1. For all cases not otherwise noted the two plotting positions give a good representation of the generated data. The complete storage data for each case run are available upon request from the authors on 7-track CDC magnetic tape (a nominal charge will be assessed to cover copying and handling costs).

4.3.2 Model Comparisons - Set 1

Figure 4.4 shows the results of seven different sets of model parameters for $D^* = 0.9$. Figures 4.5 and 4.6 contain similar information for $D^* = 0.7$

Table 4.1. Summary of Theoretical Extreme Value Type I Probability Distributions Describing Storage Needs (FFGN and Lag-one Markov Generation)

GENERATOR PARAMETERS				DEMAND = 0.9μ		DEMAND = 0.7μ		DEMAND = 0.5μ	
CV	ρ	G	H	S_1	S_2	S_1	S_2	S_1	S_2
.25	.00	.00	.50	2.08	.72	.81	.24	.41	.06
.25	.20	.00	.50	2.74	.86	.99	.26	.46	.05
.25	.40	.00	.50	3.75	1.03	1.29	.29	.49	.06***
.25	.10	.00	.60	2.95	.85	.93	.25	.41	.05
.25	.20	.00	.60	3.30	.91	1.07	.26	.45	.05***
.25	.40	.00	.60	4.38	1.11	1.4	.29	.49	.05***
.25	.20	.00	.70	3.99	.94	1.11	.24	.41	.04
.25	.40	.00	.70	5.11	1.13	1.56	.28	.53	.05***
.25	.45	.00	.70	5.44	1.23	1.66	.29	.53	.05***
.25	.45	.00	.85	6.29	1.06*	1.54	.19		**
.25	.60	.00	.85	7.88	1.33	2.06	0.24*	0.49	0.02***
.25	.20	.75	.50	2.30	.75	0.50	0.14		**
.25	.45	.75	.85	10.69	2.62*	4.30	0.96	1.48	0.34
.35	.20	.00	.50	4.70	1.48	1.94	.58	1.00	.24
.35	.45	.00	.85	11.88	1.91*	4.90	.58*	1.71	.17***
.50	.00	.00	.50	5.77	1.96	2.52	0.94	1.36	.52
.50	.20	.00	.50	7.13	2.33	3.34	1.11	1.74	.57
.50	.40	.00	.50	9.56	2.95	4.76	1.43	2.48	.70
.50	.10	.00	.60	7.95	2.28	3.70	1.08	1.80	.56
.50	.20	.00	.60	8.78	2.53	4.06	1.19	2.02	.59
.50	.20	.00	.70	10.60	2.68	4.83	1.20	2.19	.58
.50	.40	.00	.60	10.71	2.89	5.34	1.36	2.67	.65
.50	.40	.00	.70	14.88	3.17	6.87	1.50*	3.20	.69
.50	.40	.00	.80	16.84	3.39*	8.62	1.51*	3.88	.66*
.50	.45	.00	.85	16.64	3.11*	8.30	1.35*	3.74	.56*
.50	.20	1.00	.50	6.73	2.21	2.70	.87	1.17	.35
.50	.40	1.00	.50	8.56	2.74	3.69	1.09	1.55	.40
.50	.20	1.00	.70	14.04	3.04	6.03	1.10	2.13	.33
.50	.40	1.00	.70	12.18	3.16	5.30	1.20	1.97	.40
.50	.40	1.00	.80	5.22	.87*	.63	.08		**
.50	.20	2.00	.50	6.06	2.01	1.78	.59	.46	.12
.50	.40	2.00	.50	7.85	2.46	2.53	.72	.61	.13
.50	.20	2.00	.70	9.70	2.52	2.91	.68	.57	.12
.50	.40	2.00	.70	10.45	2.74	3.51	.79	.74	.12
.75	.20	2.00	.50	11.15	3.71	5.20	1.67	2.22	.70
.75	.40	2.00	.50	13.16	4.40	6.67	2.05	2.93	.85
.75	.20	3.00	.50	10.12	3.53	4.00	1.32	1.26	.41
.75	.40	3.00	.50	12.83	4.23	5.85	1.72	1.81	.51
1.00	.20	3.00	.50	11.32	3.87	5.87	1.97	2.75	.92
1.00	.40	3.00	.50	14.95	4.93	8.42	2.63	4.03	1.22
1.00	.20	3.00	.70	17.83	4.59	9.74	2.35	4.29	1.03
1.00	.40	3.00	.70	19.94	5.40	11.38	2.84	5.36	1.27

S_1 is the theoretical storage corresponding to cumulative probability 0.995

S_2 is the theoretical storage corresponding to cumulative probability 0.500

* Theoretical extreme value distribution did not adequately describe the empirical storage distribution.

** Not a predominantly over-year storage case.

*** Theoretical distribution did not fully describe the empirical distribution partly because the case was not strictly an over-year storage situation.

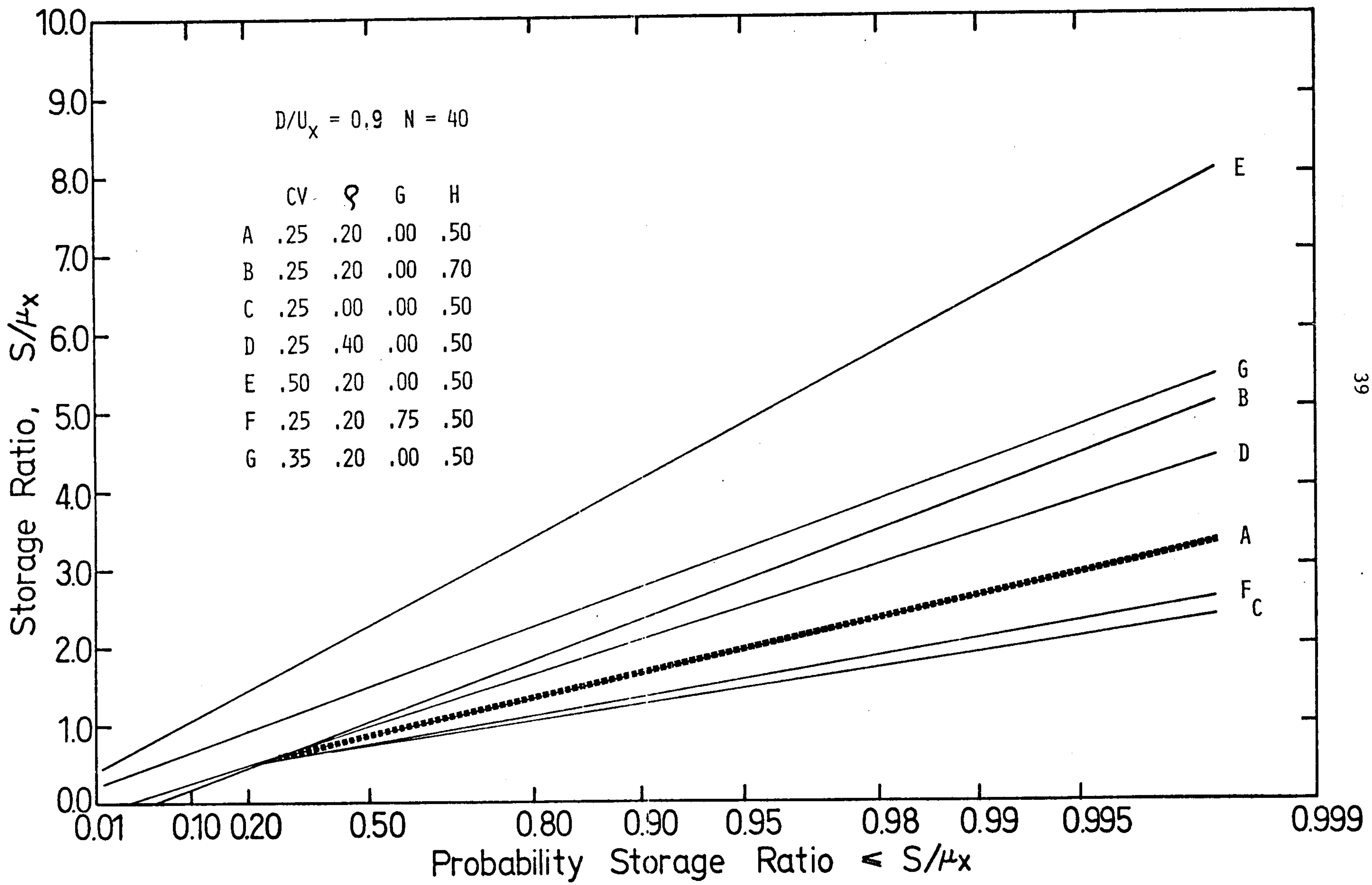


Figure 4.4. Comparison of Flow Generation Models: Set No 1, Theoretical Storage Probability Distributions; Demand/Mean Flow = .9.

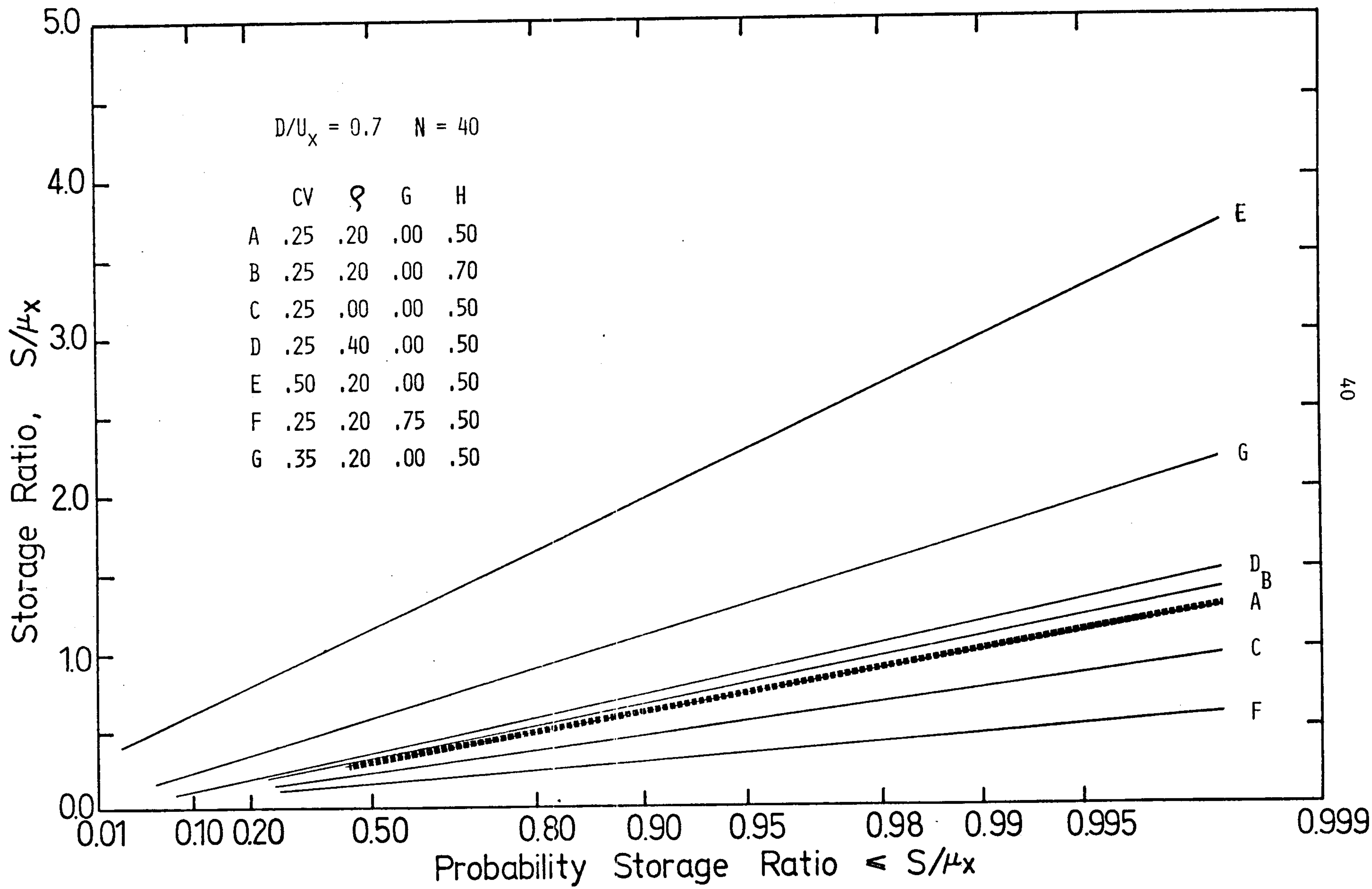


Figure 4.5. Comparison of Flow Generation Models: Set No 1, Theoretical Storage Probability Distributions; Demand/Mean Flow = 0.7.

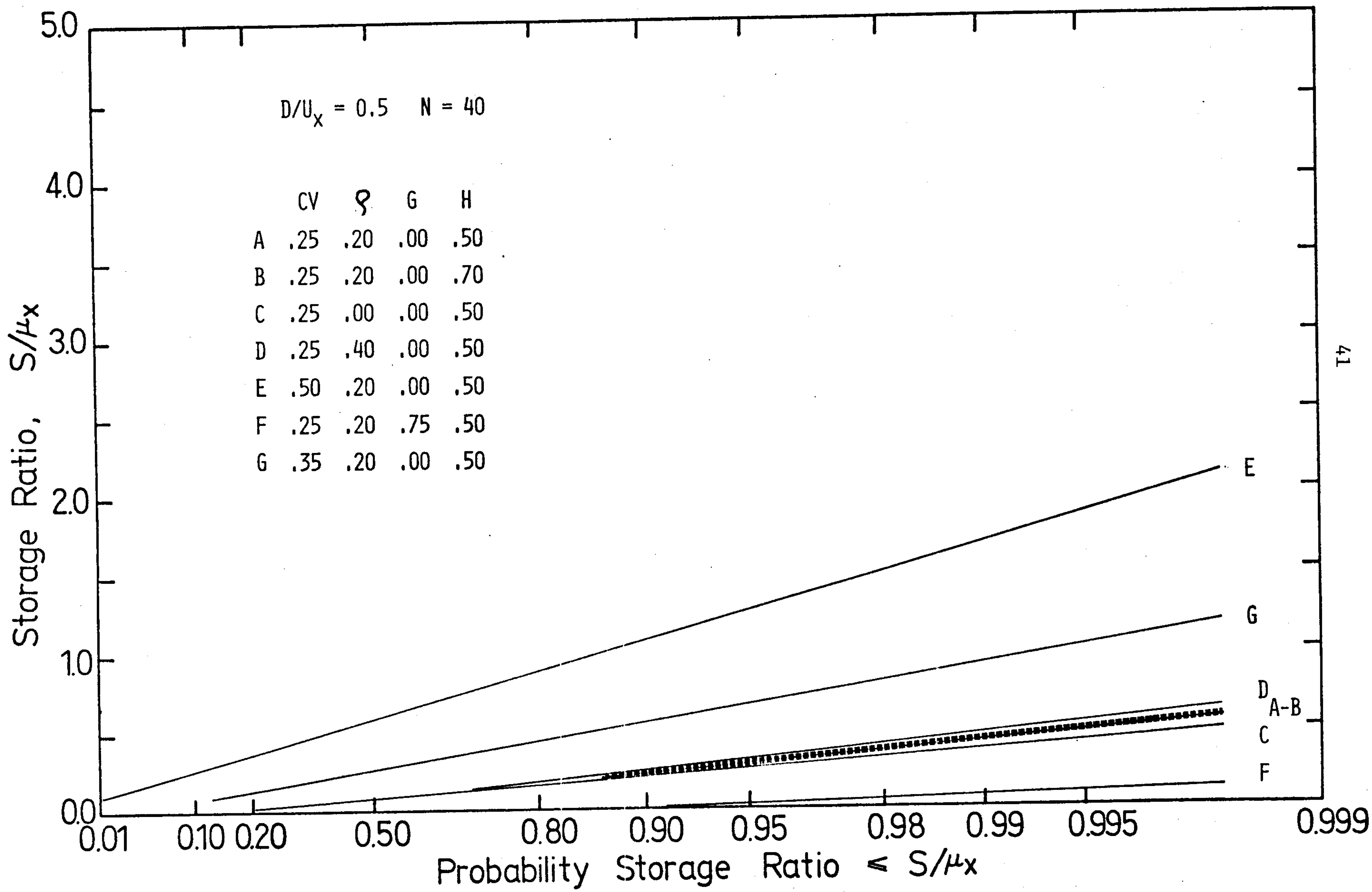


Figure 4.6. Comparison of Flow Generation Models: Set No 1, Theoretical Storage Probability Distributions; Demand/Mean Flow = 0.5.

and 0.5, respectively. Figure 4.4 shows the importance of coefficient of variation (CV), lag one correlation coefficient (ρ), skew coefficient (G) and Hurst coefficient (H). For H = 0.5, ρ = 0 and G = 0, storage needs at any given reliability level are seen to increase dramatically as CV increases from 0.25 to 0.5. The lower value of CV is representative of many snowmelt-fed U.S. streams as well as streams in humid climates. The larger value of CV is representative (but is nowhere near the upper limit) of streams which exhibit larger variability in annual flow. The importance of knowledge of ρ (compare curves A and C) is apparent. Curves A and B show the relative impact of H on the distribution of required storage. There is not an appreciable difference between the two curves at the 50% reliability (or probability) level, but substantial differences occur at higher reliability levels. Curve A corresponds to H = 0.5, curve B to H = 0.7, where all other parameters were held constant. Earlier work by Matalas and Wallis (1972) concentrated on the "average storage", i.e. the storage corresponding to the 57% reliability level for the extreme value distribution. In practice much larger reliabilities (eg. greater than ∿90%) are usually required, hence the emphasis on the complete probability-storage plots in this report.

For the parameter levels in Figure 4.4 it appears that uncertainty in H and CV relative to the base level (curve A) are about equally important. This result is, of course, dependent upon the magnitude of the changes considered in each parameter, for instance, a change of 0.15 in CV is approximately equivalent to a change of 0.2 in H about the base level of Figure 4.4. Curves A and F show the importance of skew in the marginal distribution. This result, and a number of other cases examined, clearly showed that positively skewed marginal distributions give rise to lesser storage needs at typical operational reliability levels than would be required for flows generated from identical models having symmetric marginal distributions.

Figures 4.5 and 4.6 (plotted to a larger scale than Figure 4.4) contain essentially the same information as Figure 4.4 but for lesser demand levels. Several of the lower curves of Figure 4.6 do not exclusively belong to the over-year class of storage problems. An interesting result in Figures 4.5 and 4.6 ($D^* = 0.7$ and 0.5, respectively) is that uncertainty in ρ is more important than uncertainty in H. The importance of CV and skew (parameters of the marginal distribution) is much more important than H and ρ (measures of persistence) for low demand levels, suggesting that the design demand level should dictate the relative importance of modelling accurately long-term persistence.

4.3.3 Model Comparisons - Set 2:

While Figures 4.4 - 4.6 show perturbations about a Markov domain, Figures 4.7 - 4.9 give graphical comparisons of FFGN storage needs. The base parameters for the FFGN models are: CV = 0.5, $\rho = 0.4$, G = 1.0, and H = 0.7. Examination of Figure 4.7 ($D^* = 0.9$) and Figure 4.9 ($D^* = 0.5$) shows that the demand level is of considerable importance when long-term persistence is significant. At the 98% reliability level (a level that many practitioners use to represent "firm yield") curve A has storage requirements that change from $S^* = 0.5$ to $S^* = 1.6$ as D^* changes from 0.9 to 0.5. This represents a change from the impractical to the potentially practical design capacity of an over-year storage facility. Reservoirs planned for streams having flow characteristics similar to those in Figure 4.7 would have to be impracticably large to support high demands, eg. $D^* \sim 0.9$ with reasonable reliability (even achieving a reliability of 80% would require a storage between $S^* = 4.5$ and 6.0).

It is clear in Figure 4.7 that increasing H to 0.8 and including positive skew yields approximately the same result as neglecting skew (G = 0) and using H = 0.7 (curves B and D). As the demand level is reduced the importance of skewed marginal distributions again becomes clearer (Figures 4.8 and 4.9).

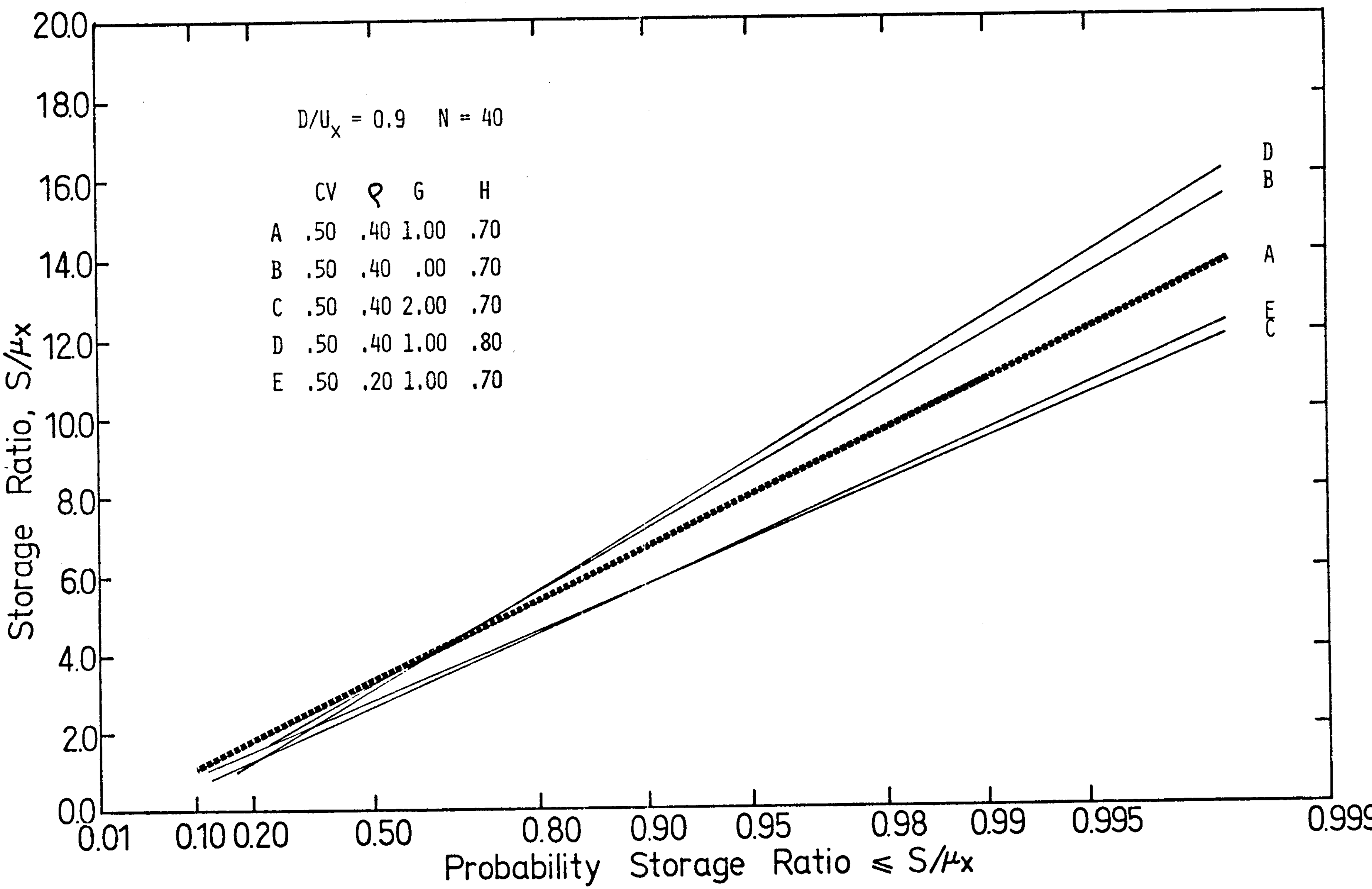


Figure 4.7. Comparison of Flow Generation Models: Set No 2, Theoretical Storage Probability Distributions, Demand/Mean Flow = 0.9.

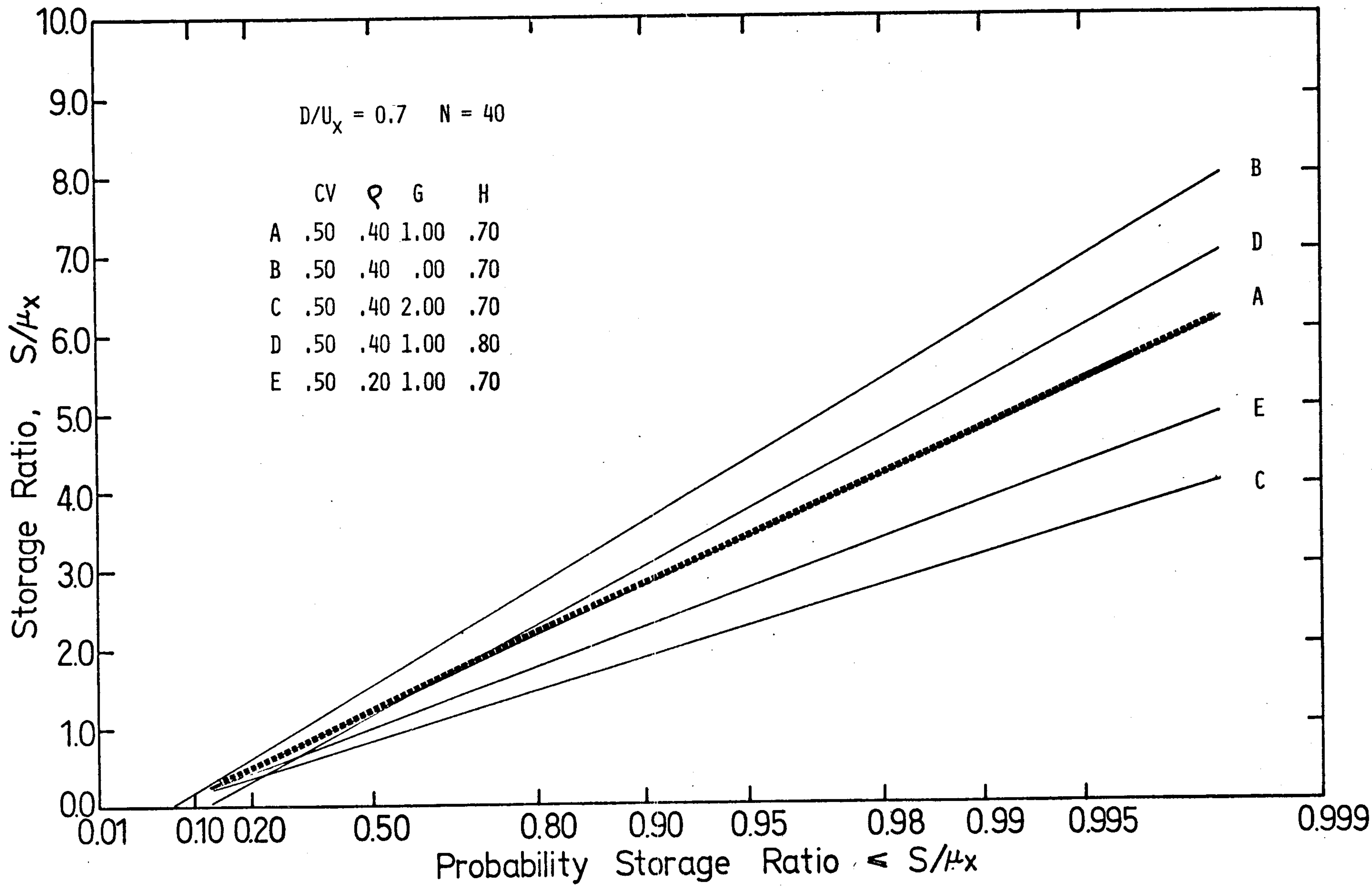


Figure 4.8. Comparison of Flow Generation Models: Set No 2, Theoretical Storage Probability Distributions; Demand/Mean Flow = 0.7.

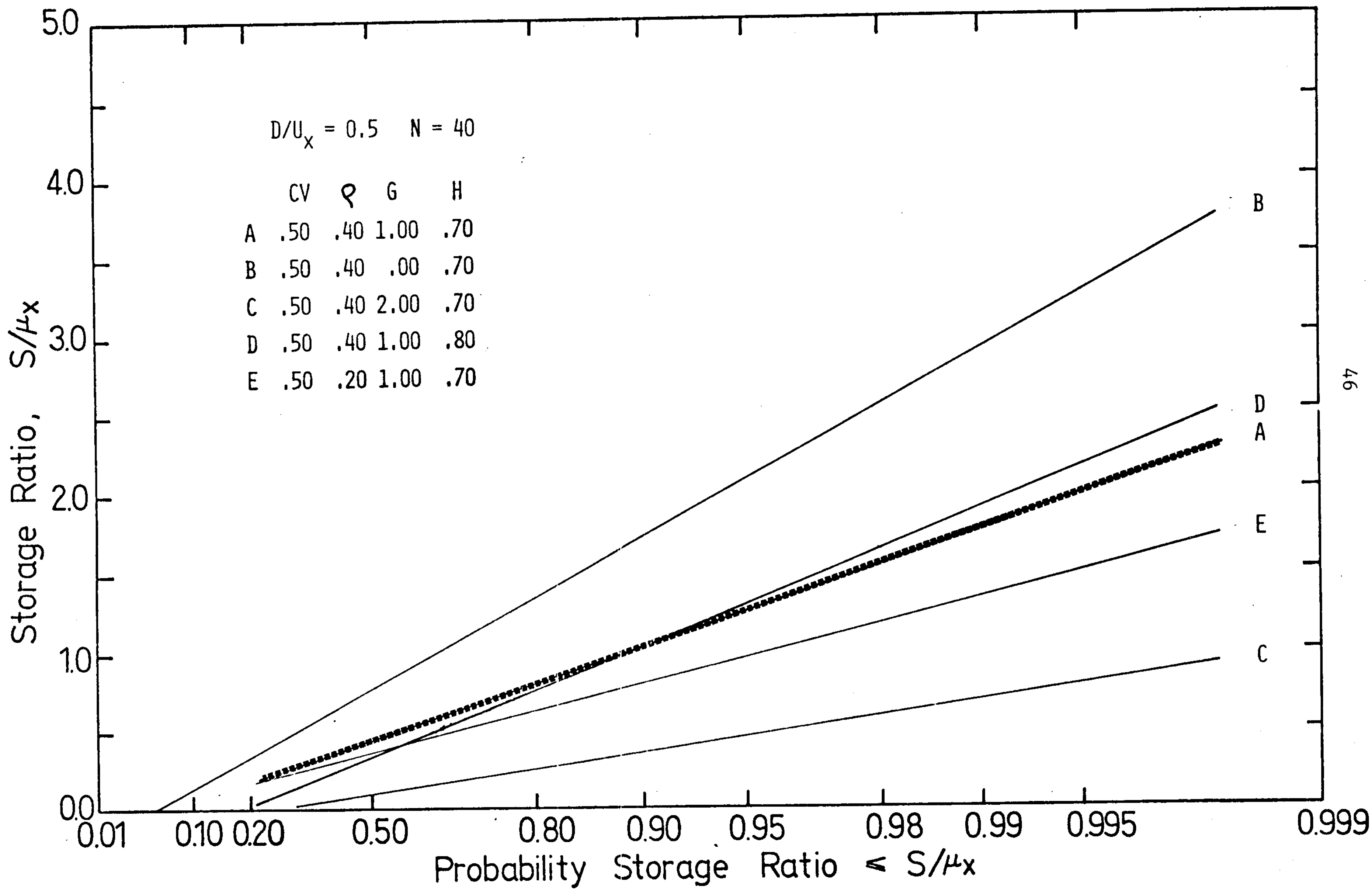


Figure 4.9. Comparison of Flow Generation Models: Set No 2, Theoretical Storage Probability Distributions, Demand/Mean Flow = 0.5.

Little attention has been focused by researchers investigating long-term persistence effects (H) upon the relative importance of ρ and G in flow modeling. The importance of uncertainty in ρ is, however, quite clear from Figures 4.7 - 4.9. It should be noted that $\rho = 0.4$ is an approximate upper limit to the lag-one correlation coefficient usually found in annual stream flow data. The FFGN model is unfortunately constrained by the feasible range of ρ and H given in Figure 2.2. At high values of H, especially, this may impose a severe practical limitation on the use of FFGN.

4.3.4 Model Comparisons - Set 3:

Figures 4.10 - 4.12 show comparisons of storages yielded from FFGN sequences. The baseline used was for $H = 0.85$ (curve A). In all cases the extreme value distribution yielded a poor fit below about the 40% reliability level. The straight lines shown do approximate the experimental data for higher reliability levels. Knowledge of CV is shown to be of utmost importance for this group of parameters. Sensitivity to ρ was much less than in sections 4.3.2 and 4.3.3 where base levels of ρ and H were lower.

4.4 <u>Observations: Critical Periods</u>

Summary statistics of the critical period information generated are given in Tables 4.2 - 4.6. For illustrative purposes properties of six (6) sets of generated flow traces are discussed in this section. The storage-probability distributions for $D^* = 0.9$ for these six cases are given in Figure 4.13. The six cases cover a range of H between 0.5 and 0.85, CV between 0.25 and 0.5, and ρ between 0.0 and 0.45. Only one case having non-zero skew coefficient is presented here. Other relevant comparisons can be drawn from Tables 4.2 - 4.6.

4.4.1 Frequency Diagrams - Number of Critical Periods:

Figures 4.14 and 4.15 show frequency diagrams of the number of critical periods per 40-year long-flow trace conditioned by demands $D^* = 0.9$ and 0.5,

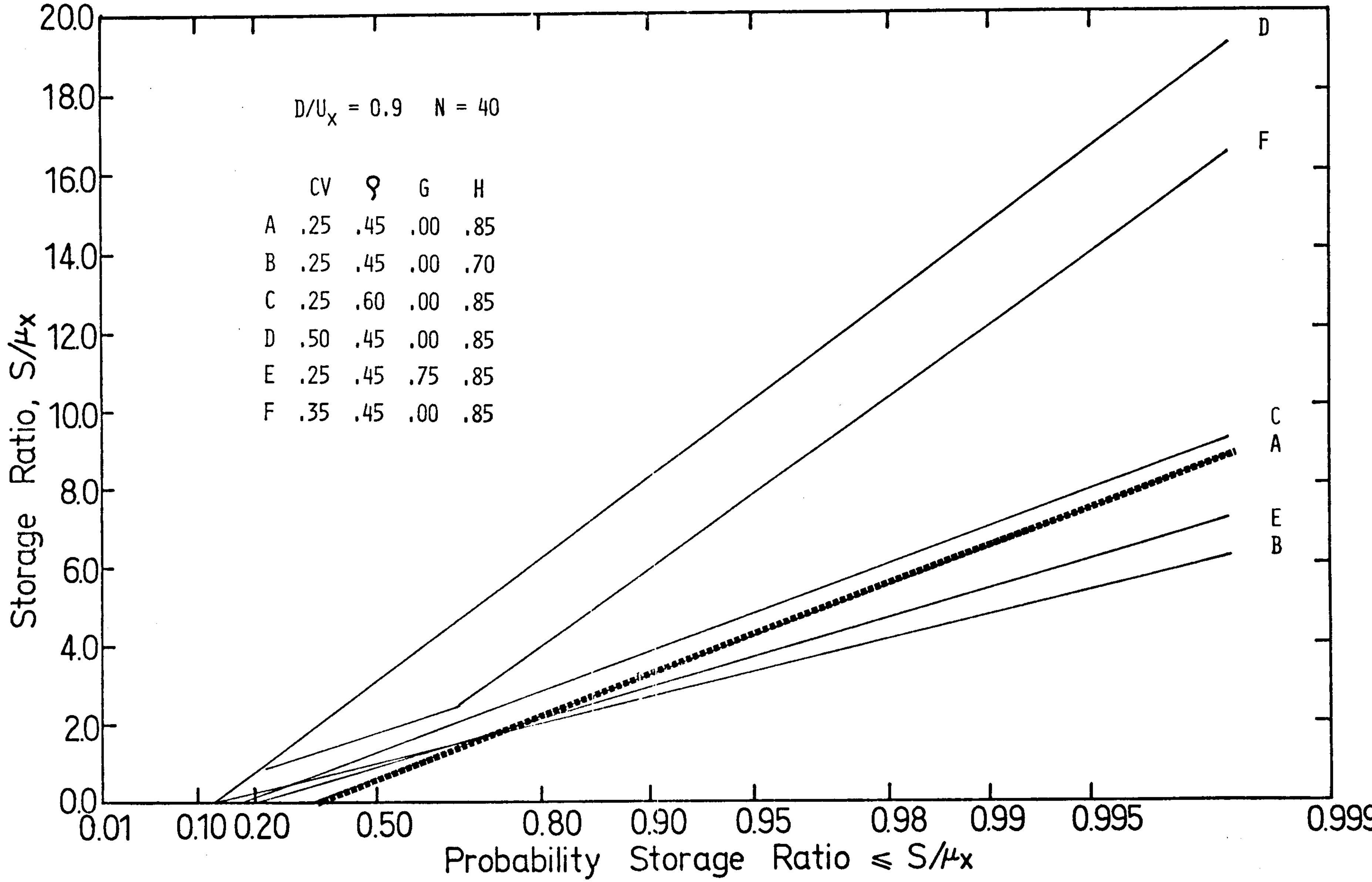


Figure 4.10. Comparison of Flow Generation Models: Set No 3, Theoretical Storage Probability Distributions; Demand/Mean Flow = 0.9.

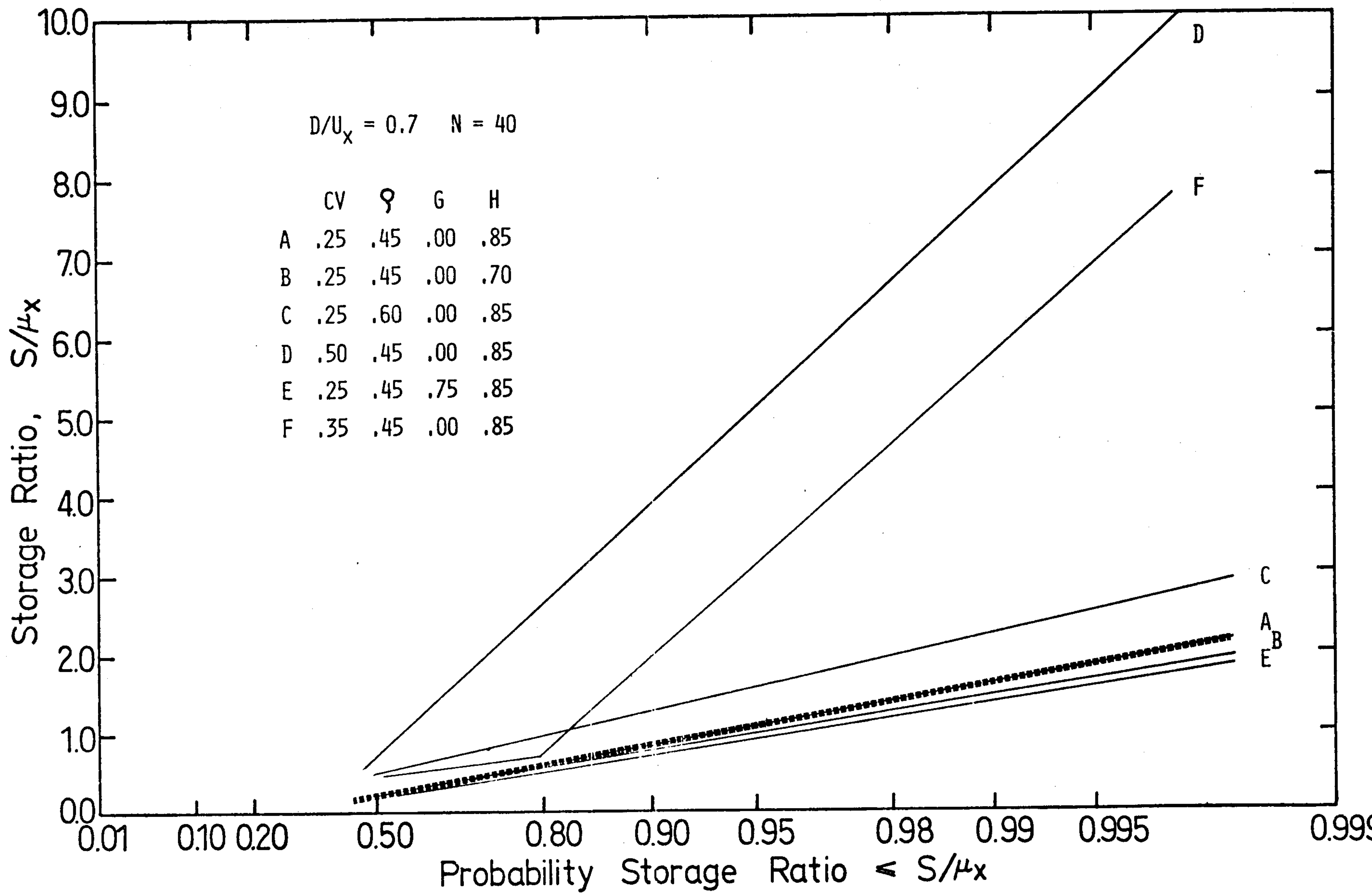


Figure 4.11. Comparison of Flow Generation Models: Set No 3, Theoretical Storage Probability Distributions; Demand/Mean Flow = 0.7.

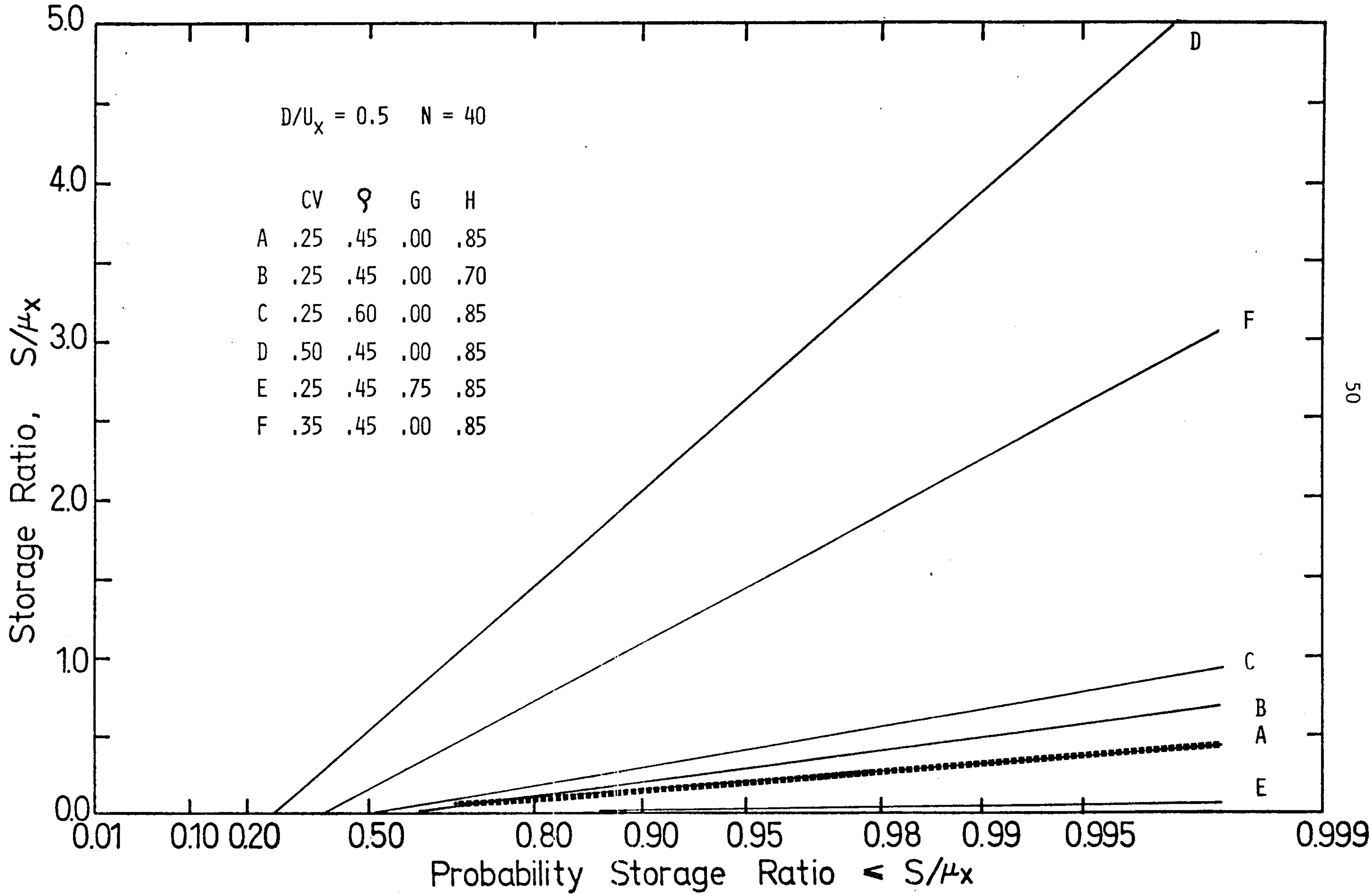


Figure 4.12. Comparison of Flow Generation Models: Set No 3, Theoretical Storage Probability Distributions; Demand/Mean Flow = 0.5.

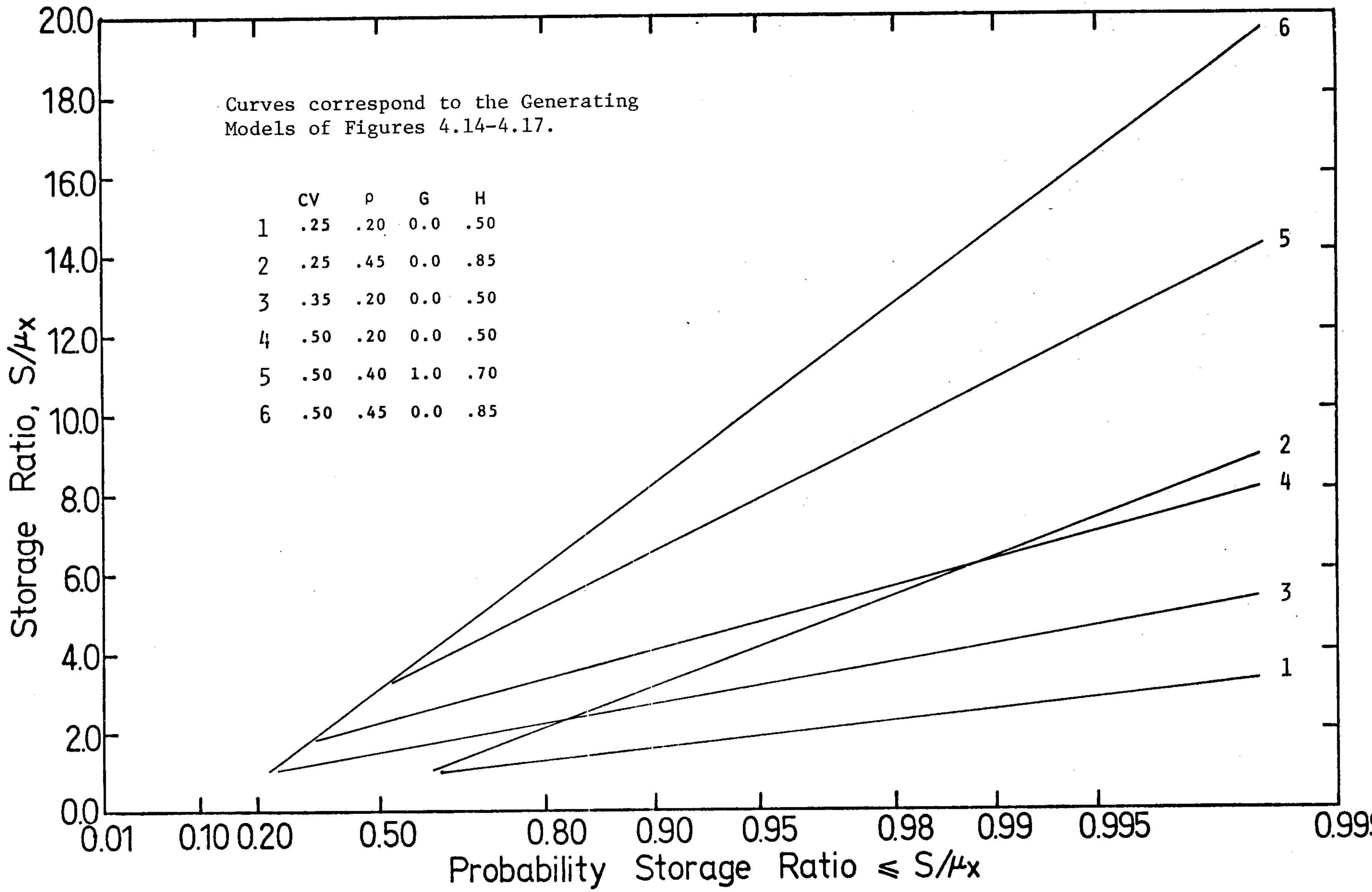
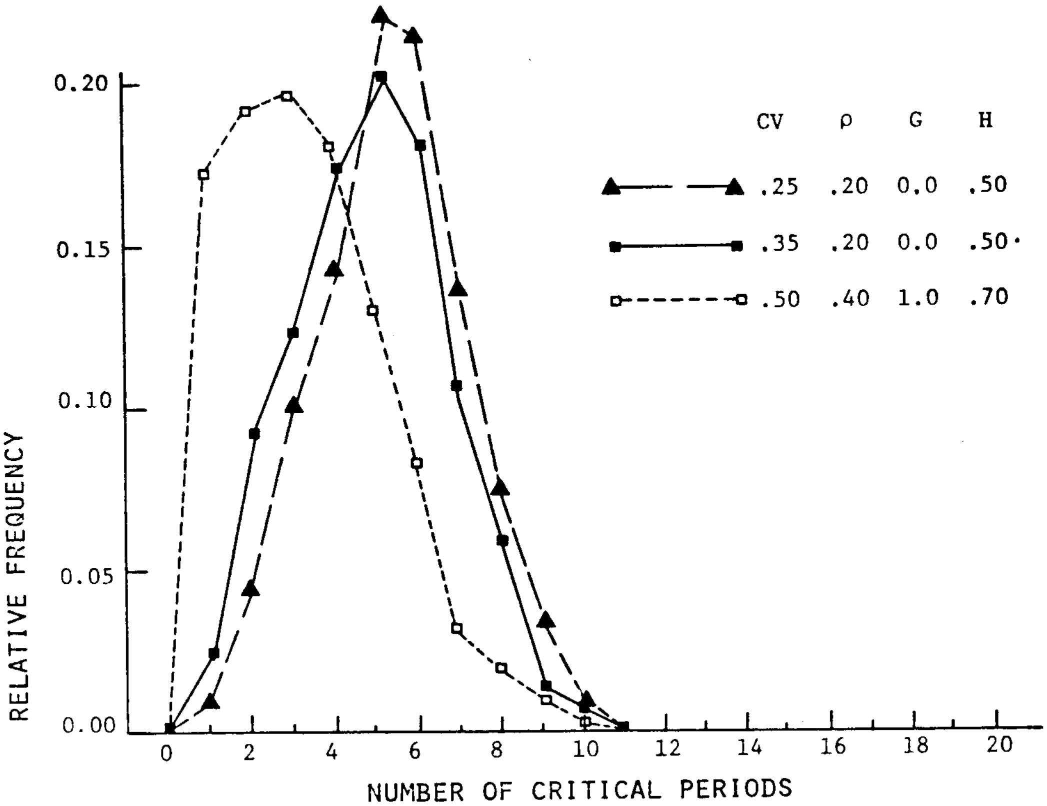


Figure 4.13. Theoretical Storage Probability Distributions; Demand/Mean Flow = 0.9.



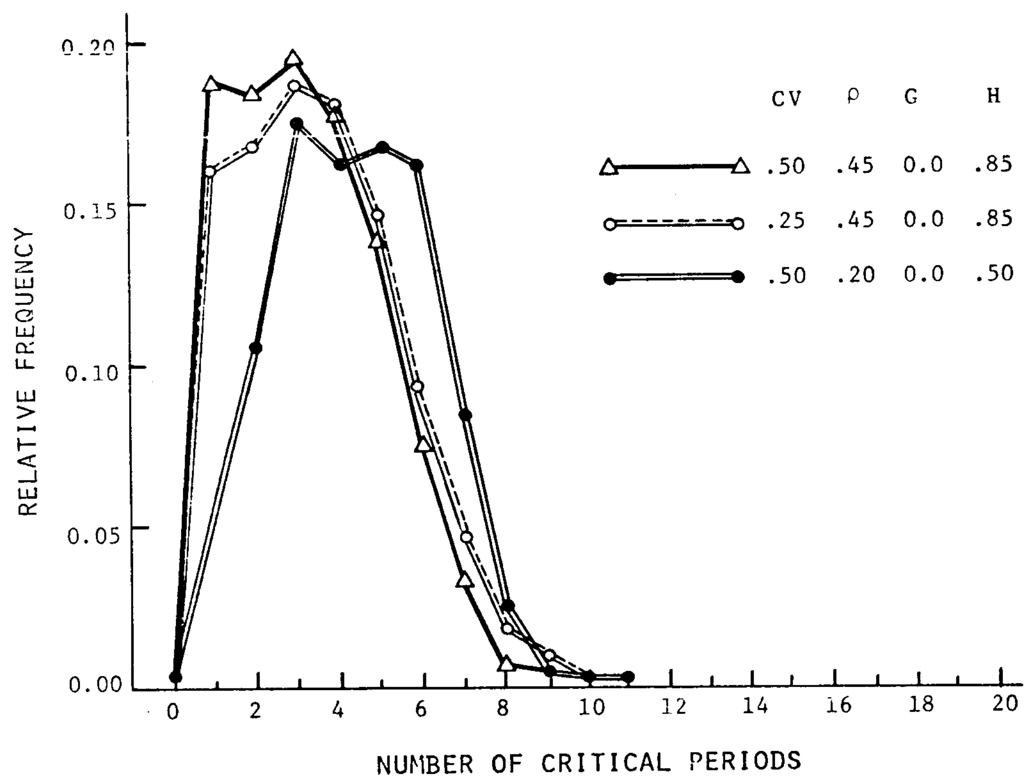
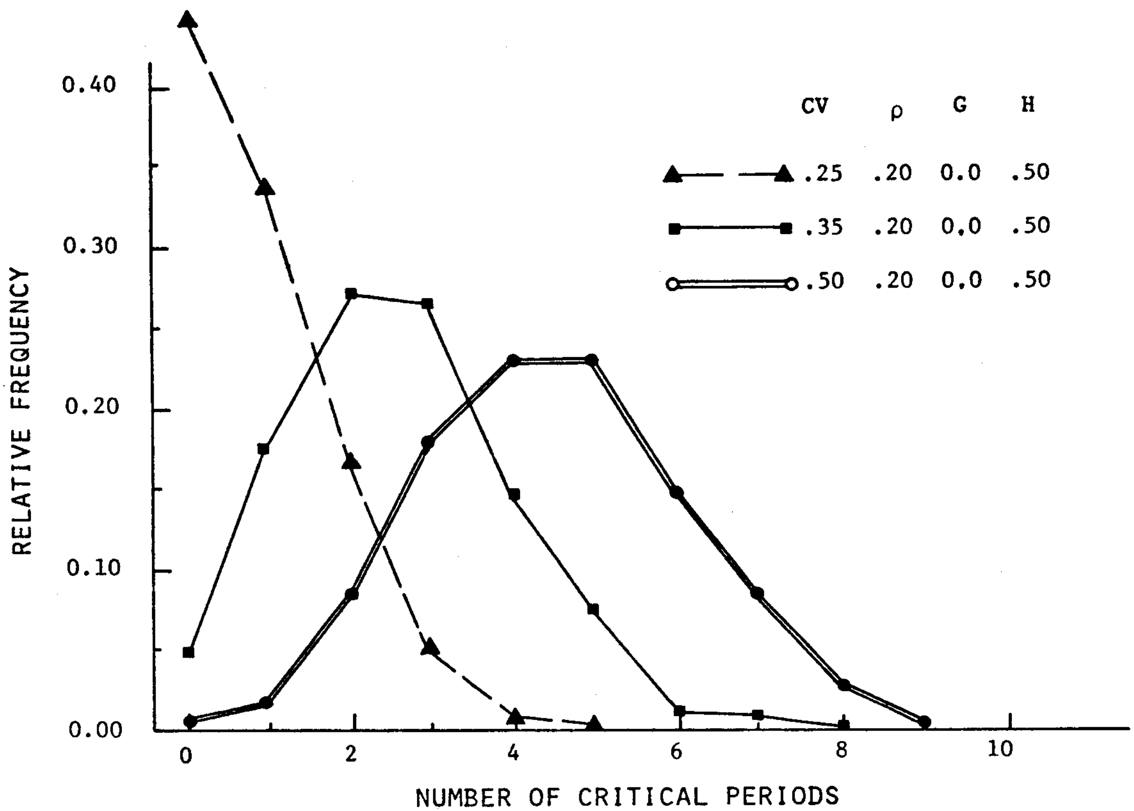


Figure 4.14. Frequency Diagram of the Number of Critical Periods per 40 Year Flow Trace; Demand/Mean Flow = 0.9.



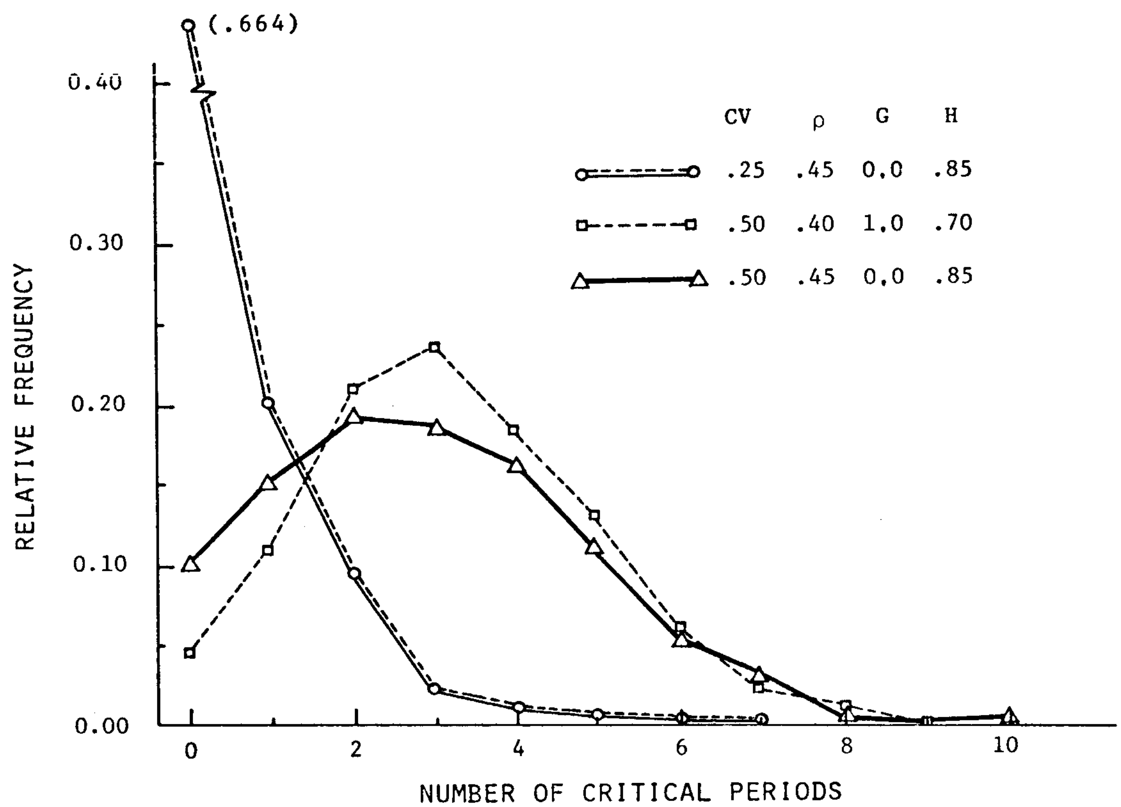


Figure 4.15. Frequency Diagram of the Number of Critical Periods per 40 Year Flow Trace; Demand/Mean Flow = 0.5.

respectively. Figure 4.14 shows, as intuitively expected, that there tend to be generally fewer critical periods for FGN sequences than for Markov sequences. Figure 4.15 qualitatively shows the same trends.

The combination of low CV and large H gives rise to a problem that is only a partial over-year storage case (particularly when $D^* \sim 0.5$). Note that the large numbers of zero critical periods correspond to small over-year storage needs. Figures 4.14 and 4.15 are useful for assessing the applicability of an extreme value model of storage. Parameter combinations involving H in the vicinity of 0.5 appear to approximately (qualitatively) satisfy the theory while those with large H ($H \gtrsim 0.8$) usually will not.

4.4.2 Frequency Diagrams - Length of Most Severe Critical Period:

Frequency diagrams for the length of the most severe critical period per trace are given in Figure 4.16 ($D^* = 0.9$) and Figures 4.17a and b ($D^* = 0.5$). This information provides another way of examining the flows generated by different models and model parameters. While these frequency diagrams are not precise they do permit qualitative comparison of the different flow generation models. Figure 4.16 has been arbitrarily truncated at a 20-year critical period length. Figure 4.16 shows that critical periods for $D^* = 0.9$ are typically between 2 and 8 years long. Combinations of low CV and high H are, however, associated with longer length critical periods.

Figures 4.17a and b show the impact of demand level upon critical period length. Regardless of the magnitude of H the duration of the longest critical period (at $D^* = 0.5$) is most commonly between 2 and 4 years with a modal value of 2 years. The high H flows do, however, give rise to a few very long critical periods even at this relatively low demand level.

Frequency distribution plots for refill times and the distribution of the longest critical period per trace were not prepared because they appeared to yield little additional information. Examination of much of our data showed

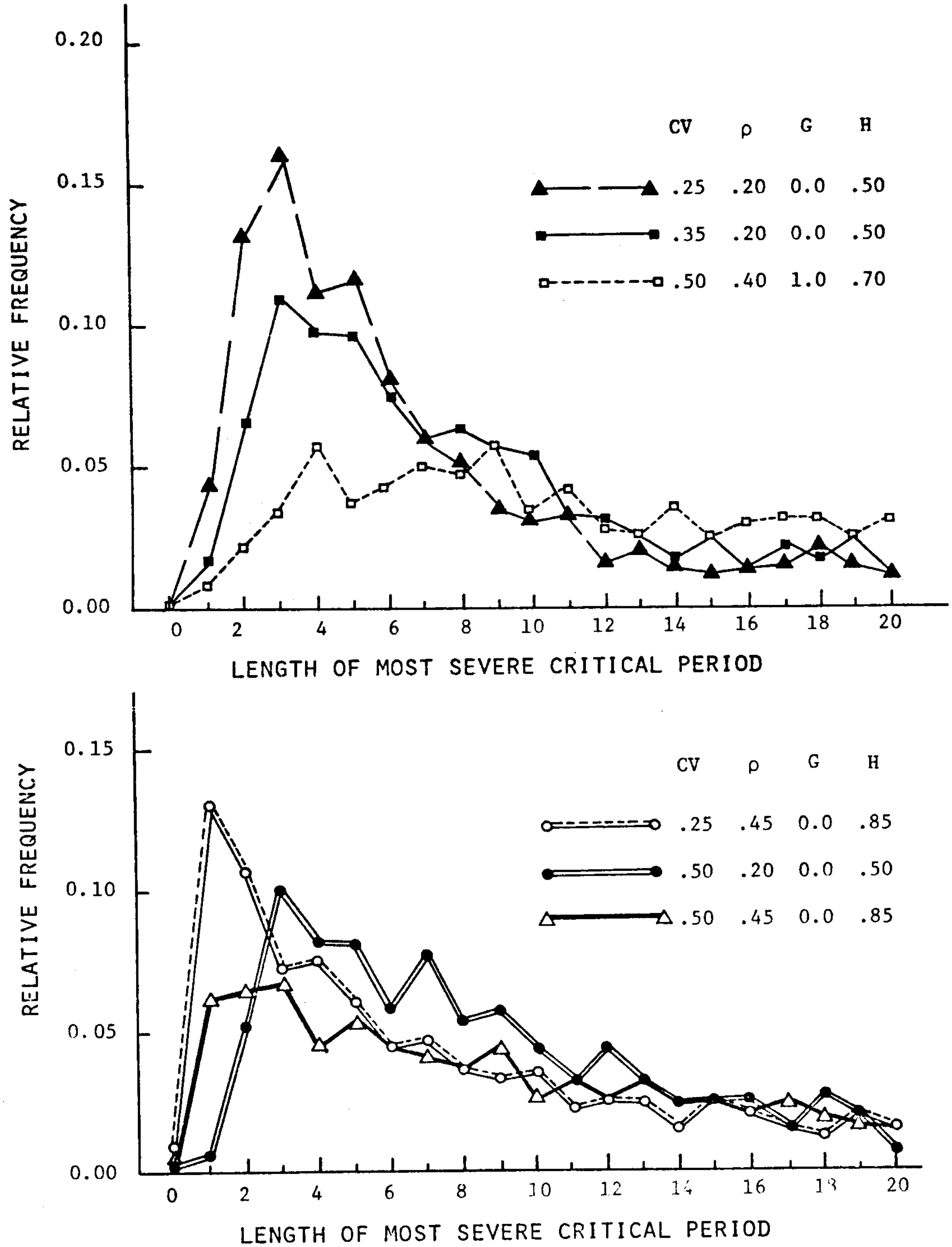


Figure 4.16. Frequency Diagram of the Length of the Most Severe Critical Period per 40 Year Trace; Demand/Mean Flow = 0.9.

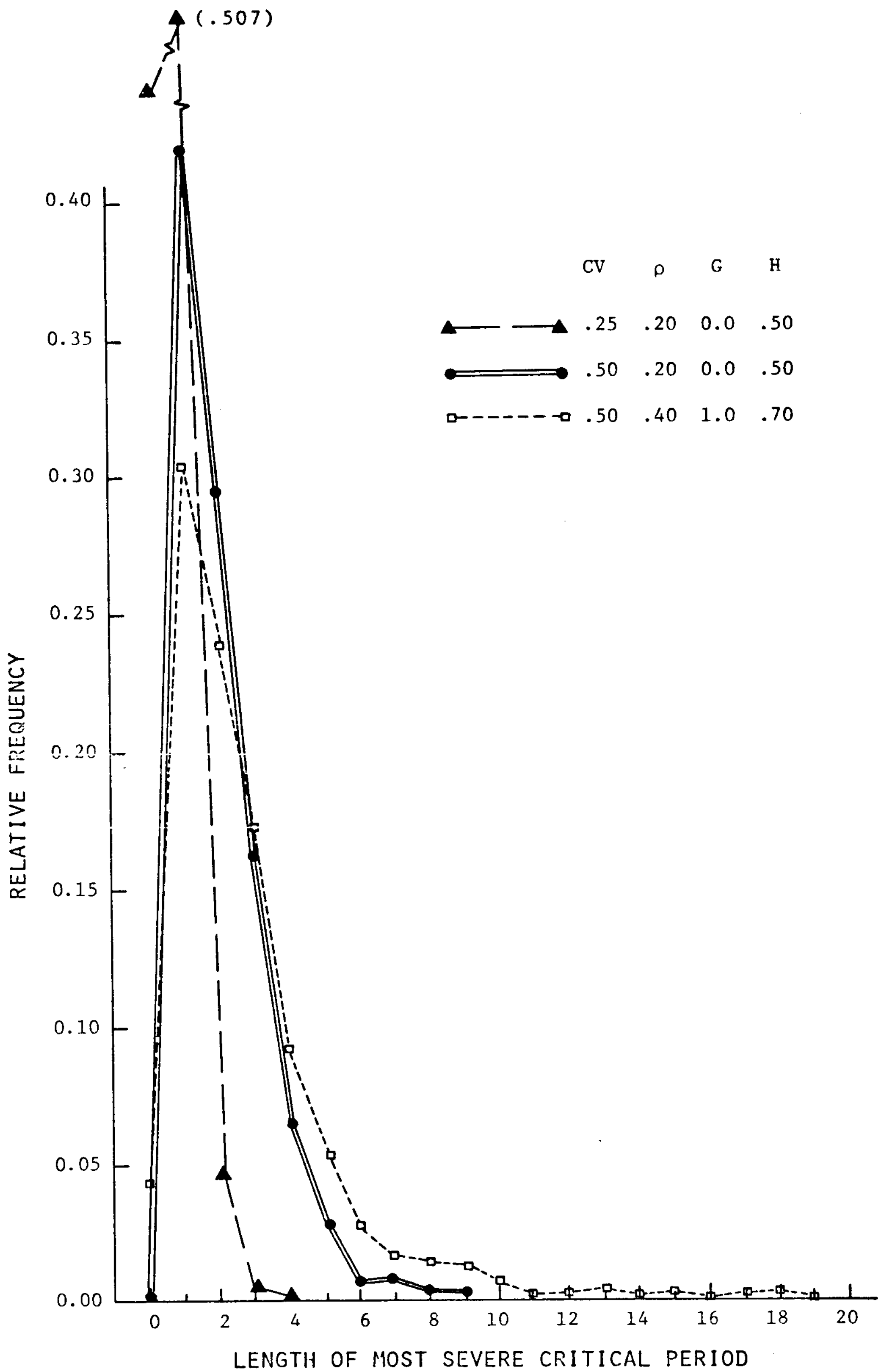


Figure 4.17a. Frequency Diagram of the Length of the Most Severe Critical Period per 40 Year Trace; Demand/Mean Flow = 0.5.

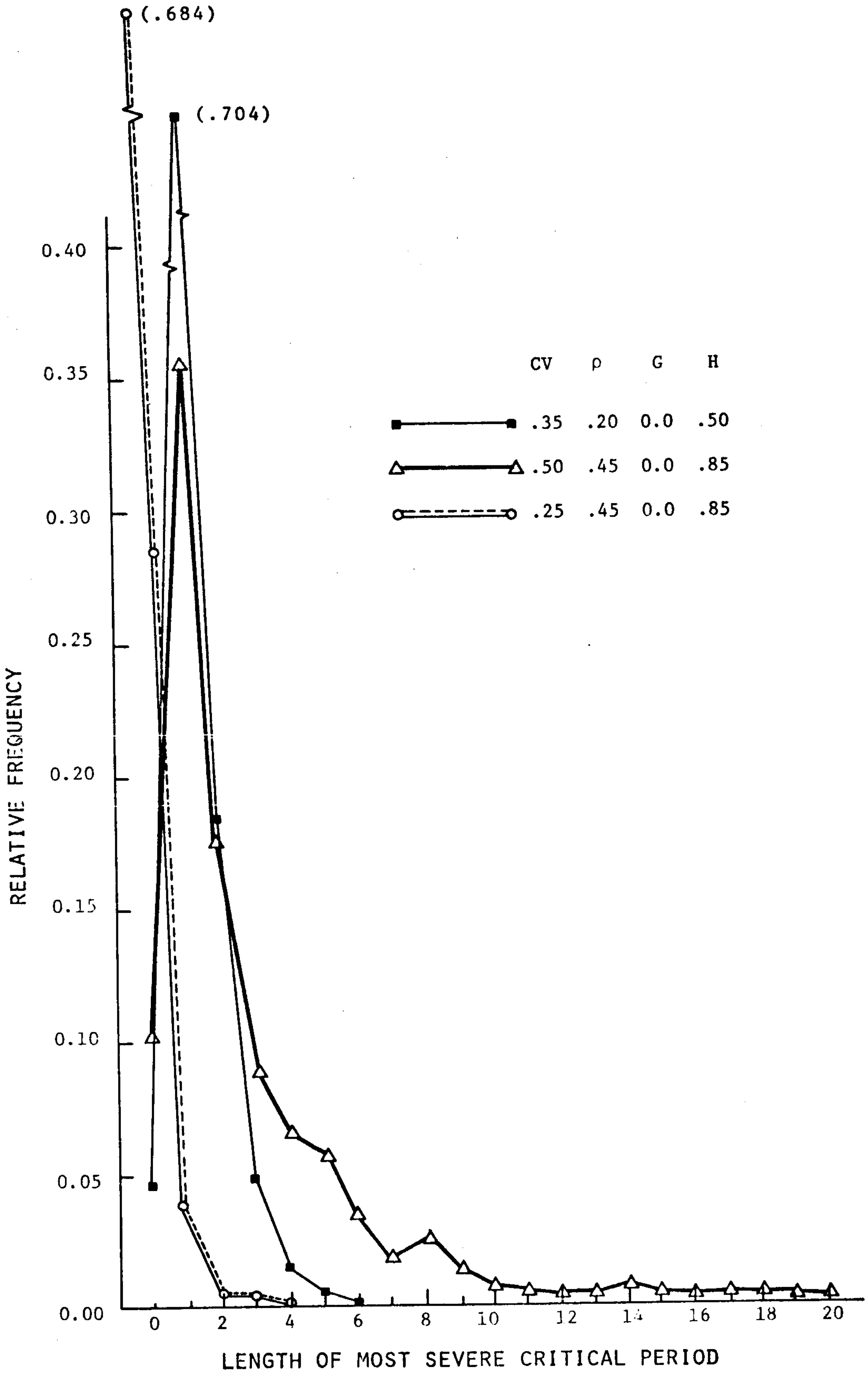


Figure 4.17b. Frequency Diagram of the Length of the Most Severe Critical Period per 40 Year Trace; Demand/Mean Flow = 0.5.

that the most severe critical period usually coincided with the longest critical period per trace. (The slight differences in the two quantities are evidenced in Tables 4.2 and 4.4.)

4.4.3 Critical Periods - Summary Statistics:

Summary statistics for the critical period data that were generated are given in Tables 4.2 - 4.6. Each table is arranged in increasing order of magnitude of CV, ρ , G and H for the convenience of the reader. The mean, standard deviation and skew coefficient of each quantity for demand levels $D^* = 0.9$, 0.7 and 0.5 provide a convenient method for condensing the voluminous data that were generated. Each statistic was computed from a sample of 1,000 independent observations.

4.4.3 (a) Most Severe Critical Period:

Table 4.2 shows that generally, as flow variability increases, the mean length of the most severe critical period increases as does its standard deviation while the skew coefficient becomes smaller. The mean critical period lengths range from 5.5 to 17.6 years for $D^* = 0.9$ and from 0.6 to 6.6 years for $D^* = 0.5$ (Critical period lengths were computed as integers; decimal quantities in the summary statistics result from necessary arithmetic operations on these integer quantities.) Table 4.2 clearly shows the enormous impact of demand level upon the duration of a critical period. It should be noted that larger mean values of critical period duration could be expected if the operation life exceeded the 40 years used in this report.

4.4.3 (b) Refill Time Following Most Severe Critical Period:

Table 4.3 summarizes refill time data. In many cases the refill time was the difference between the time the reservoir would have emptied and year 40. In such cases, while the refill algorithm used considers the reservoir to have filled, in fact it would still be partially empty at year 40. Despite this

Table 4.2. Summary Statistics from 1000 Traces for the Length of the Most Severe Critical Period per 40 Year Long Trace for FFGN and Markov Models.

GENERATOR PARAMETERS				LENGTH OF MOST SEVERE CRITICAL PERIOD DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
.25	.00	.00	.50	5.51	4.40	1.87	1.36	.70	2.31	.63	.53	-.04
.25	.20	.00	.50	6.65	5.45	1.88	1.63	1.03	2.26	.62	.60	.55
.25	.40	.00	.50	7.90	6.12	1.53	2.05	1.45	1.99	.69	.74	1.27
.25	.10	.00	.60	6.98	6.21	1.91	1.60	1.08	2.27	.63	.55	.18
.25	.20	.00	.60	7.40	6.35	1.74	1.75	1.25	2.29	.63	.61	.58
.25	.40	.00	.60	8.80	7.13	1.56	2.21	1.85	4.20	.64	.77	1.67
.25	.20	.00	.70	8.66	7.83	1.44	1.77	1.42	2.44	.56	.63	1.13
.25	.40	.00	.70	9.70	8.24	1.31	2.25	2.17	3.13	.66	.85	2.01
.25	.45	.00	.85	10.56	10.37	1.27	2.10	3.00	3.74	.42	.76	4.56
.25	.60	.00	.85	12.37	11.27	.98	2.73	3.64	3.21	.51	1.18	8.70
.25	.20	.75	.50	7.66	5.78	1.60	1.46	.82	1.59	.15	.36	2.26
.25	.45	.75	.85	16.21	9.42	.64	7.76	5.55	1.79	3.51	2.27	1.67
.35	.20	.00	.50	9.08	6.80	1.41	2.66	1.91	2.26	1.30	.76	2.23
.35	.45	.00	.85	12.87	11.43	.94	4.37	6.15	2.84	1.51	2.58	5.23
.50	.00	.00	.50	9.70	7.38	1.34	3.18	2.33	2.19	1.68	.99	2.06
.50	.20	.00	.50	11.01	7.89	1.15	4.09	3.17	2.41	2.08	1.31	1.94
.50	.40	.00	.50	12.79	8.47	.92	5.55	4.06	2.00	2.85	1.92	1.78
.50	.10	.00	.60	10.85	7.76	1.06	4.16	3.65	2.83	2.08	1.63	4.27
.50	.15	.00	.65	11.74	9.39	1.14	4.36	4.18	2.81	2.05	1.58	3.55
.50	.20	.00	.60	11.87	8.79	1.09	4.76	3.88	2.10	2.23	1.56	2.35
.50	.20	.00	.70	13.15	10.11	.93	5.25	5.05	2.49	2.41	2.25	3.74
.50	.40	.00	.60	12.77	9.18	.99	5.62	4.65	2.09	2.86	2.35	2.51
.50	.40	.00	.75	13.63	10.44	.90	6.25	6.40	2.23	2.93	3.01	3.00
.50	.40	.00	.80	15.16	11.84	.66	7.60	8.09	1.77	3.40	4.16	3.59
.50	.45	.00	.85	14.55	11.86	.75	7.04	7.87	1.92	3.22	4.27	3.32
.50	.20	1.00	.50	12.31	8.09	1.00	4.24	3.00	2.05	1.88	1.20	2.25
.50	.40	1.00	.50	14.11	8.89	.79	5.58	3.92	1.97	2.39	1.61	1.84
.50	.20	1.00	.70	16.43	11.70	.56	7.19	7.38	2.01	2.64	3.02	3.62
.50	.40	1.00	.70	11.01	7.89	1.15	4.09	3.17	2.41	2.08	1.31	1.94
.50	.40	1.00	.80	12.79	8.47	.92	5.55	4.06	2.00	2.85	1.92	1.78
.50	.20	2.00	.50	13.12	7.90	.89	4.24	2.73	1.72	1.39	.79	1.54
.50	.40	2.00	.50	15.61	9.12	.68	5.44	3.95	1.86	1.69	1.22	1.93
.50	.20	2.00	.70	16.31	10.66	.63	5.73	5.10	2.09	1.54	1.21	2.19
.50	.40	2.00	.70	16.89	10.57	.56	6.52	5.50	2.08	1.80	1.62	2.35
.75	.20	2.00	.50	16.21	9.42	.64	7.76	5.55	1.79	3.51	2.27	1.67
.75	.40	2.00	.50	17.63	9.31	.51	7.78	5.34	1.71	2.96	1.77	1.72
.75	.20	3.00	.50	17.63	9.31	.51	7.78	5.34	1.71	2.96	1.77	1.72
.75	.40	3.00	.50	18.75	9.71	.41	10.22	7.00	1.34	3.97	2.68	1.96
1.00	.20	3.00	.50	13.97	8.44	.87	7.87	5.29	1.64	4.11	2.63	1.94
1.00	.40	3.00	.50	15.96	8.83	.67	9.95	6.36	1.22	5.78	3.96	2.10
1.00	.20	3.00	.70	16.10	10.56	.66	10.07	8.25	1.43	5.42	4.72	2.19
1.00	.40	3.00	.70	17.64	10.63	.46	11.44	8.56	1.21	6.60	5.51	1.87

Table 4.3. Summary Statistics from 1000 Traces for Reservoir Refill Time Following the Most Severe Critical Period per 40 Year Long Trace, for FFGN and Markov Models.

GENERATOR PARAMETERS				REFILL TIME AFTER MOST SEVERE CRITICAL PERIOD								
				DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
.25	.00	.00	.50	5.74	4.34	1.86	1.53	.81	1.67	.64	.55	.10
.25	.20	.00	.50	6.60	5.10	1.64	1.72	1.04	2.31	.63	.62	.64
.25	.40	.00	.50	7.37	5.71	1.55	2.00	1.34	2.08	.69	.72	1.05
.25	.10	.00	.60	6.36	4.80	1.51	1.70	1.05	2.11	.68	.63	.61
.25	.20	.00	.60	6.84	5.21	1.38	1.86	1.33	2.53	.65	.63	.81
.25	.40	.00	.60	7.85	6.05	1.37	2.14	1.56	2.61	.66	.78	1.43
.25	.20	.00	.70	7.10	6.00	1.62	1.86	1.44	2.99	.58	.68	1.19
.25	.40	.00	.70	7.48	5.98	1.38	2.24	2.04	3.88	.67	.84	1.76
.25	.45	.00	.85	6.53	6.14	1.58	2.04	2.50	3.26	.42	.74	3.31
.25	.60	.00	.85	6.74	6.04	1.31	2.55	3.25	3.03	.49	.91	3.02
.25	.20	.75	.50	5.92	4.81	1.96	1.38	.79	2.27	.14	.36	2.21
.25	.45	.75	.85	8.81	6.76	1.07	4.78	4.16	2.28	2.34	1.64	2.03
.35	.20	.00	.50	8.36	5.89	1.29	2.72	1.90	2.53	1.37	.74	1.39
.35	.45	.00	.85	7.02	6.35	1.29	3.58	3.98	2.54	1.35	1.84	4.03
.50	.00	.00	.50	8.72	6.15	1.14	3.07	2.01	1.55	1.63	.92	1.92
.50	.20	.00	.50	9.08	6.43	1.20	3.75	2.53	1.79	1.93	1.17	2.31
.50	.40	.00	.50	9.99	6.81	.88	4.99	3.63	1.80	2.46	1.50	1.54
.50	.10	.00	.60	8.86	6.66	1.23	3.74	2.82	2.26	1.90	1.22	2.54
.50	.15	.00	.65	8.57	6.70	1.12	4.18	3.57	2.22	1.92	1.23	2.03
.50	.20	.00	.60	8.90	6.29	1.14	4.28	3.44	2.41	2.14	1.46	2.44
.50	.20	.00	.70	7.92	6.14	1.31	4.66	4.00	2.29	2.17	1.50	1.96
.50	.40	.00	.60	9.02	6.73	1.07	5.03	4.16	2.12	2.41	1.85	5.41
.50	.40	.00	.75	8.55	7.04	1.18	5.33	4.72	1.94	2.61	2.45	3.12
.50	.40	.00	.80	7.56	6.38	1.19	5.57	5.30	1.82	2.92	3.05	3.05
.50	.45	.00	.85	7.34	6.60	1.27	5.24	5.38	1.98	2.80	3.21	2.84
.50	.20	1.00	.50	8.48	6.34	1.07	3.56	3.03	3.05	1.63	.92	1.93
.50	.40	1.00	.50	8.69	6.48	1.10	4.29	3.39	2.16	2.03	1.35	2.06
.50	.20	1.00	.70	7.09	6.21	1.16	4.99	4.87	2.04	2.12	2.24	3.98
.50	.40	1.00	.70	9.08	6.43	1.20	3.75	2.53	1.79	1.93	1.17	2.31
.50	.40	1.00	.80	9.99	6.81	.88	4.99	3.63	1.80	2.46	1.50	1.54
.50	.20	2.00	.50	7.99	6.30	1.29	2.67	2.00	2.33	1.21	.58	1.62
.50	.40	2.00	.50	8.36	6.30	1.12	3.54	2.90	2.42	1.39	.90	2.36
.50	.20	2.00	.70	7.75	6.38	1.14	3.79	3.64	2.58	1.32	.97	3.42
.50	.40	2.00	.70	7.28	5.89	1.34	4.16	3.75	2.39	1.43	1.16	2.86
.75	.20	2.00	.50	8.81	6.76	1.07	4.78	4.16	2.28	2.34	1.64	2.03
.75	.40	2.00	.50	7.97	6.47	1.18	4.52	3.77	1.77	1.97	1.35	2.57
.75	.20	3.00	.50	7.97	6.47	1.18	4.52	3.77	1.77	1.97	1.35	2.57
.75	.40	3.00	.50	8.05	6.35	1.03	5.35	4.30	1.83	2.59	1.91	2.27
1.00	.20	3.00	.50	7.13	5.54	1.24	4.38	3.70	2.01	2.51	1.78	1.81
1.00	.40	3.00	.50	8.44	6.53	1.12	5.68	4.53	1.83	3.33	2.45	1.93
1.00	.20	3.00	.70	5.70	5.79	1.44	5.28	4.83	1.79	3.21	3.14	2.96
1.00	.40	3.00	.70	6.90	5.85	1.33	5.98	5.31	1.76	3.86	3.57	2.38

qualification it can be seen that flows generated via models having positively skewed marginal distributions give rise to more rapid refill times than those having zero skew. Generally, recovery times were of the same order as drawdown times but were considerably shorter for highly variable highly skewed flows.

Tables 4.4 and 4.5 essentially replicate Tables 4.2 and 4.3 for the case of the longest, rather than the most severe critical period. These tables indicate that the most severe and the longest critical period per trace are often but not always the same. Thus when using the SPA no distinction appears necessary between severity and length of critical periods.

4.4.3 (c) Number of Critical Periods

Table 4.6 summarizes the number of critical periods per 40-year trace. In some instances the number of critical periods correspond to an 80-year period because of the two-cycle computation scheme used in the SPA (Fiering, 1967). Generally the flows generated from low CV and high H yield fewer critical periods than those from highly variable and positively skewed models for a fixed value of D^*. The average number of critical periods per trace ranged from 6.46 to 2.77 for $D^* = 0.9$, while the corresponding range was 0.94 to 0.50 for $D^* = 0.5$. Table 4.6 clearly shows the impact of CV, ρ, G, and H on the number of critical periods per trace, particularly for $D^* = 0.5$. The mean number of critical periods per trace shown for the three demand levels illustrates the interaction of demand level, trace variability and persistence. The importance of "Noah" and "Joseph" type events is quite clear for the case: CV = 0.5, ρ = 0.4, G = 1.0, H = 0.8. Compare this, for example with the case: CV = 0.5, ρ = 0.4, G = 0 and H = 0.80. This comparison emphasizes the importance of the skew of the marginal distribution, particularly at lower demand levels. Many other interesting comparisons can be found in Table 4.6.

Table 4.4. Summary Statistics from 1000 Traces for the Length of the Longest Critical Period per 40 Year Long Trace for FFGN and Markov Models.

GENERATOR PARAMETERS				LENGTH OF LONGEST CRITICAL PERIOD								
				DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
.25	.00	.00	.50	6.16	4.18	1.93	1.56	.79	1.59	.63	.54	.03
.25	.20	.00	.50	7.15	5.27	1.92	1.81	1.06	1.85	.62	.61	.56
.25	.40	.00	.50	8.37	5.91	1.55	2.19	1.46	1.83	.70	.75	1.26
.25	.10	.00	.60	7.49	5.99	1.96	1.78	1.10	1.83	.64	.57	.29
.25	.20	.00	.60	7.89	6.14	1.77	1.93	1.26	1.99	.64	.63	.77
.25	.40	.00	.60	9.15	6.95	1.60	2.38	1.86	3.98	.65	.78	1.66
.25	.20	.00	.70	8.99	7.65	1.47	1.93	1.45	2.17	.56	.65	1.12
.25	.40	.00	.70	9.98	8.09	1.32	2.41	2.19	2.91	.67	.86	1.93
.25	.45	.00	.85	10.79	10.25	1.27	2.20	3.00	3.67	.43	.78	4.35
.25	.60	.00	.85	12.47	11.19	.99	2.84	3.68	3.07	.53	1.20	8.27
.25	.20	.75	.50	8.19	5.56	1.64	1.57	.85	1.26	.15	.36	2.26
.25	.45	.75	.85	16.56	9.13	.69	8.18	5.37	1.83	3.86	2.19	1.73
.35	.20	.00	.50	9.56	6.54	1.47	2.95	1.84	2.24	1.39	.80	1.79
.35	.45	.00	.85	13.08	11.29	.95	4.56	6.11	2.82	1.57	2.60	5.09
.50	.00	.00	.50	10.21	7.10	1.40	3.66	2.24	2.20	1.90	1.00	1.61
.50	.20	.00	.50	11.50	7.59	1.19	4.51	3.05	2.43	2.32	1.31	1.74
.50	.40	.00	.50	13.16	8.21	.97	5.86	3.92	2.08	3.04	1.90	1.72
.50	.10	.00	.60	11.32	7.48	1.09	4.51	3.53	2.91	2.30	1.62	4.10
.50	.15	.00	.65	12.17	9.10	1.18	4.71	4.07	2.85	2.26	1.59	3.26
.50	.20	.00	.60	12.26	8.56	1.12	5.14	3.81	2.07	2.47	1.57	2.20
.50	.20	.00	.70	13.50	9.87	.95	5.61	4.97	2.47	2.63	2.23	3.64
.50	.40	.00	.60	13.13	8.97	1.01	5.94	4.53	2.12	3.05	2.34	2.42
.50	.40	.00	.75	13.88	10.25	.93	6.57	6.32	2.22	3.12	3.01	2.90
.50	.40	.00	.80	15.33	11.70	.68	7.87	7.99	1.76	3.55	4.16	3.53
.50	.45	.00	.85	14.73	11.73	.76	7.19	7.81	1.93	3.36	4.24	3.29
.50	.20	1.00	.50	12.80	7.75	1.06	4.69	2.89	2.04	2.10	1.21	2.03
.50	.40	1.00	.50	14.54	8.62	.82	5.95	3.84	1.94	2.57	1.59	1.70
.50	.20	1.00	.70	16.61	11.54	.58	7.49	7.31	1.99	2.76	3.01	3.57
.50	.40	1.00	.70	11.50	7.59	1.19	4.51	3.05	2.43	2.32	1.31	1.74
.50	.40	1.00	.80	13.16	8.21	.97	5.86	3.92	2.08	3.04	1.90	1.72
.50	.20	2.00	.50	13.53	7.61	.95	4.59	2.64	1.72	1.50	.84	1.23
.50	.40	2.00	.50	15.92	8.88	.71	5.81	3.85	1.83	1.78	1.25	1.79
.50	.20	2.00	.70	16.58	10.44	.66	6.08	5.03	2.06	1.65	1.25	1.91
.50	.40	2.00	.70	17.10	10.40	.58	6.78	5.42	2.10	1.88	1.66	2.21
.75	.20	2.00	.50	16.56	9.13	.69	8.18	5.37	1.83	3.86	2.19	1.73
.75	.40	2.00	.50	17.94	9.04	.56	8.20	5.15	1.77	3.24	1.72	1.69
.75	.20	3.00	.50	17.94	9.04	.56	8.20	5.15	1.77	3.24	1.72	1.69
.75	.40	3.00	.50	19.00	9.48	.44	10.59	6.81	1.37	4.25	2.64	1.91
1.00	.20	3.00	.50	14.38	8.16	.91	8.30	5.16	1.62	4.44	2.53	1.99
1.00	.40	3.00	.50	15.25	8.58	.72	10.32	6.19	1.23	6.04	3.86	2.15
1.00	.20	3.00	.70	16.45	10.32	.67	10.39	8.07	1.46	5.72	4.67	2.17
1.00	.40	3.00	.70	17.89	10.42	.48	11.74	8.41	1.22	6.88	5.48	1.84

Table 4.5. Summary Statistics from 1000 Traces for Reservoir Refill Time Following the Longest Critical Period per 40 Year Long Trace for FFGN and Markov Models.

GENERATOR PARAMETERS				REFILL TIME AFTER LONGEST CRITICAL PERIOD								
				DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
.25	.00	.00	.50	5.22	4.38	1.93	1.36	.69	2.14	.63	.53	-.06
.25	.20	.00	.50	6.02	5.17	1.72	1.55	1.00	2.81	.62	.61	.59
.25	.40	.00	.50	6.85	5.65	1.58	1.80	1.26	2.46	.68	.70	1.07
.25	.10	.00	.60	5.88	4.91	1.57	1.53	.96	2.43	.66	.60	.50
.25	.20	.00	.60	6.36	5.26	1.43	1.67	1.25	2.99	.64	.62	.79
.25	.40	.00	.60	7.50	6.15	1.39	2.04	1.58	2.70	.64	.75	1.36
.25	.20	.00	.70	6.62	5.87	1.64	1.68	1.38	3.51	.56	.64	1.13
.25	.40	.00	.70	7.11	6.07	1.38	2.11	2.06	4.01	.65	.83	1.82
.25	.45	.00	.85	6.18	6.09	1.62	1.93	2.49	3.43	.42	.74	3.49
.25	.60	.00	.85	6.59	6.07	1.33	2.46	3.22	3.13	.49	.90	3.09
.25	.20	.75	.50	5.44	4.62	1.92	1.27	.72	2.97	.14	.36	2.21
.25	.45	.75	.85	8.43	6.73	1.10	4.54	4.07	2.26	2.12	1.55	2.24
.35	.20	.00	.50	7.77	5.98	1.25	2.39	1.90	2.78	1.26	.66	1.67
.35	.45	.00	.85	6.76	6.35	1.34	3.32	3.97	2.67	1.29	1.80	4.25
.50	.00	.00	.50	8.14	6.18	1.14	2.69	1.96	1.71	1.46	.82	2.36
.50	.20	.00	.50	8.64	6.51	1.23	3.36	2.59	1.87	1.77	1.15	2.67
.50	.40	.00	.50	9.62	6.79	.87	4.66	3.70	1.82	2.26	1.47	1.67
.50	.10	.00	.60	8.42	6.68	1.20	3.42	2.83	2.37	1.74	1.18	2.79
.50	.15	.00	.65	7.98	6.67	1.17	3.81	3.53	2.22	1.77	1.23	2.48
.50	.20	.00	.60	8.34	6.32	1.13	3.98	3.49	2.49	1.93	1.42	2.80
.50	.20	.00	.70	7.59	6.18	1.33	4.32	3.85	2.03	2.00	1.49	2.19
.50	.40	.00	.60	8.55	6.74	1.11	4.69	4.20	2.16	2.27	1.86	5.48
.50	.40	.00	.75	8.12	6.91	1.20	5.13	4.85	1.90	2.45	2.44	3.21
.50	.40	.00	.80	7.31	6.37	1.17	5.32	5.32	1.83	2.76	3.00	3.07
.50	.45	.00	.85	7.07	6.52	1.26	5.09	5.38	1.99	2.66	3.22	2.89
.50	.20	1.00	.50	8.05	6.39	1.12	3.13	2.78	3.37	1.50	.84	2.24
.50	.40	1.00	.50	8.30	6.45	1.12	3.94	3.27	2.13	1.87	1.30	2.27
.50	.20	1.00	.70	6.95	6.24	1.19	4.74	4.75	1.97	2.05	2.25	4.02
.50	.40	1.00	.70	8.64	6.51	1.23	3.36	2.59	1.87	1.77	1.15	2.67
.50	.40	1.00	.80	9.62	6.79	.87	4.66	3.70	1.82	2.26	1.47	1.67
.50	.20	2.00	.50	7.74	6.29	1.31	2.46	1.96	2.46	1.15	.51	1.48
.50	.40	2.00	.50	8.04	6.16	1.15	3.29	2.74	2.33	1.31	.85	2.72
.50	.20	2.00	.70	7.53	6.39	1.16	3.56	3.58	2.61	1.27	.94	3.79
.50	.40	2.00	.70	7.17	5.90	1.33	3.98	3.80	2.39	1.39	1.14	3.00
.75	.20	2.00	.50	8.43	6.73	1.10	4.54	4.07	2.26	2.12	1.55	2.24
.75	.40	2.00	.50	7.64	6.27	1.20	4.26	3.70	1.87	1.83	1.31	2.76
.75	.20	3.00	.50	7.64	6.27	1.20	4.26	3.70	1.87	1.83	1.31	2.74
.75	.40	3.00	.50	7.95	6.32	1.04	5.15	4.36	1.85	2.43	1.91	2.45
1.00	.20	3.00	.50	6.84	5.53	1.24	4.14	3.65	2.08	2.31	1.72	2.06
1.00	.40	3.00	.50	8.08	6.44	1.15	5.46	4.47	1.78	3.14	2.43	1.98
1.00	.20	3.00	.70	6.50	5.72	1.46	5.11	4.76	1.76	3.04	3.03	2.98
1.00	.40	3.00	.70	6.82	5.77	1.31	5.78	5.25	1.75	3.71	3.54	2.41

Table 4.6. Summary Statistics from 1000 Traces for the Number of Critical Periods per 40 Year Long Trace for FFGN and Markov Models.

GENERATOR PARAMETERS				NUMBER OF CRITICAL PERIODS								
				DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
.25	.00	.00	.50	6.46	1.88	-.07	4.03	1.63	.28	.94	.96	.87
.25	.20	.00	.50	5.40	1.81	.06	3.52	1.55	.29	.84	.92	.93
.25	.40	.00	.50	4.42	1.67	.20	3.14	1.48	.34	.83	.89	.93
.25	.10	.00	.60	5.56	1.94	.07	3.71	1.70	.27	.92	.95	.96
.25	.20	.00	.60	5.09	1.79	.01	3.42	1.61	.32	.85	.92	1.05
.25	.40	.00	.60	4.10	1.68	.34	3.00	1.47	.24	.77	.92	1.17
.25	.20	.00	.70	4.72	1.92	.24	3.28	1.74	.38	.74	.92	1.32
.25	.40	.00	.70	3.82	1.64	.32	2.71	1.53	.35	.75	.93	1.30
.25	.45	.00	.85	3.55	1.89	.39	2.39	1.87	.59	.54	.93	2.30
.25	.60	.00	.85	2.77	1.49	.54	2.01	1.59	.57	.50	.86	1.99
.25	.20	.75	.50	5.49	1.82	.15	3.05	1.53	.41	.15	.40	2.52
.25	.45	.75	.85	4.03	2.12	.37	5.21	1.82	.10	3.80	1.87	.32
.35	.20	.00	.50	4.89	1.87	.07	5.17	1.60	.26	2.61	1.39	.43
.35	.45	.00	.85	3.44	1.83	.42	3.22	1.85	.32	1.65	1.59	1.01
.50	.00	.00	.50	5.34	2.13	.17	6.75	1.60	.02	5.08	1.72	.13
.50	.20	.00	.50	4.55	2.01	.30	5.90	1.63	.10	4.59	1.63	.22
.50	.40	.00	.50	3.71	1.72	.49	4.70	1.55	.28	3.91	1.50	.10
.50	.10	.00	.60	4.71	2.03	.16	5.93	1.72	.11	4.57	1.70	.21
.50	.15	.00	.65	4.65	2.25	.20	5.73	1.75	.00	4.55	1.84	.36
.50	.20	.00	.60	4.28	1.95	.22	5.43	1.66	.20	4.31	1.65	.19
.50	.20	.00	.70	4.14	2.06	.35	4.98	1.76	.15	4.13	1.84	.20
.50	.40	.00	.60	3.66	1.73	.51	4.33	1.55	.28	3.55	1.53	.30
.50	.40	.00	.75	3.66	1.88	.44	4.28	1.77	.30	3.56	1.74	.22
.50	.40	.00	.80	3.31	1.82	.52	3.74	1.75	.33	3.27	1.76	.30
.50	.45	.00	.85	3.30	1.78	.49	3.63	1.78	.28	2.94	1.91	.41
.50	.20	1.00	.50	4.58	2.03	.37	6.06	1.62	.07	4.18	1.58	.13
.50	.40	1.00	.50	3.78	1.82	.64	4.97	1.53	.36	3.52	1.49	.17
.50	.20	1.00	.70	3.39	1.87	.54	3.93	1.70	.31	2.96	1.84	.50
.50	.40	1.00	.70	3.34	1.75	.48	4.18	1.56	.13	3.20	1.67	.31
.50	.40	1.00	.80	3.71	1.86	.35	1.81	1.68	.94	.05	.25	5.04
.50	.20	2.00	.50	4.59	1.98	.20	6.10	1.63	.15	2.81	1.44	.32
.50	.40	2.00	.50	3.67	1.74	.40	5.00	1.53	.18	2.46	1.44	.54
.50	.20	2.00	.70	3.93	2.15	.49	5.30	1.78	.16	2.68	1.69	.56
.50	.40	2.00	.70	3.38	1.76	.59	4.17	1.50	.24	2.18	1.48	.55
.75	.20	2.00	.50	3.92	2.04	.59	5.87	1.79	-.08	5.93	1.61	.22
.75	.40	2.00	.50	3.30	1.72	.59	4.76	1.67	.16	4.80	1.46	.27
.75	.20	3.00	.50	3.76	1.88	.50	5.98	1.79	-.05	5.64	1.65	.06
.75	.40	3.00	.50	3.21	1.65	.61	4.59	1.64	.14	4.47	1.55	.24
1.00	.20	3.00	.50	4.65	2.03	.27	6.08	1.79	-.02	6.36	1.62	.06
1.00	.40	3.00	.50	3.60	1.70	.50	4.66	1.66	.11	4.93	1.56	.31
1.00	.20	3.00	.70	4.14	2.20	.41	5.19	2.04	-.01	5.39	1.80	.10
1.00	.40	3.00	.70	3.31	1.72	.63	3.97	1.69	.42	4.14	1.59	.16

4.5 Summary

Operational comparisons of FFGN and Markov flow generation models were made to examine the relative importance of the form of the generated flows to "project planning". An integrated index, the storage that would be needed to satisfy a given (constant) demand, was used for comparison purposes. It was convenient to make use of the Extreme Value Type I probability distribution to describe experimentally generated storage values resulting from a given generation scheme and demand pattern.

It is clear that flow populations having high H ($H \gtrsim 0.8$) cannot satisfy large demands ($D^* \gtrsim 0.7$) at any generally accepted reliability level unless infeasibly large storage facilities are used. The important implications of this result for long-term water resource planning are readily apparent.

It was found that for three parameter log normally distributed annual flows (for which the normal distribution is a special case) ρ, G, H and CV were all important when coupled with various values of D^* to determine storage needs. For $H \approx 0.7$, ρ, G and CV were extremely important in influencing storage needs. Very little attention had previously been directed towards the importance of ρ and G in flow generation schemes that modeled long term persistence (H). Because ρ is sometimes important in storage studies current versions of FFGN may have severe limitations in practical applications because of restricted feasible combinations of ρ and H (Figure 2.2).

The practical importance of obtaining the complete distribution of storage rather than a few low moment summary statistics is evident. Study of the cumulative distribution shows the importance of ρ, G, CV and H. The importance of these parameters was predicated upon use of the SPA which provides a conservative assessment of storage needs. The SPA does, however reflect the strong flow-demand coupling of interest in practice.

The importance of the parameters CV, ρ, G, and H depends on their relative levels. We have shown situations where substantial changes in ρ make little difference at all compared with the impact of H. In other situations CV is quite important relative to H and ρ. In other cases G is of considerable importance. Thus different degrees of precision are needed in estimating CV, ρ, G, and H depending on the magnitudes of the parameters and on the demand levels. Generally, however, it appears to be necessary to accurately identify CV, ρ, G and H. The economic implications of use of an improperly identified model are evident from the differences in number of critical periods and relative lengths of below demand (and presumably above demand) level flows.

5. OPERATIONAL COMPARISONS: ARMA AND FFGN MODELS

5.1 Introduction

O'Connell (1971) originally proposed the ARMA (1,1) (hereafter referred to as ARMA) model as a simple method of representing the Hurst effect. In more recent work (O'Connell, 1974) he developed the approximate expected values, based on Monte Carlo techniques, of the Hurst coefficient and lag one correlation coefficient as a function of the generation parameters, ϕ and θ, and the record length, n. The primary advantage of the ARMA model is its computational simplicity; as yet, however, a comparison of the quality of simulation yielded from the ARMA model and its more cumbersome cousin, FFGN, has not been made.

The results of Chapter 4 suggest that an effective comparison of the properties of FFGN and ARMA generated sequences can be made via the storage-probability curves resulting from use of each generator for identical demand conditions. Because our earlier results had shown that parameters other than H can be quite important in storage design, it was decided to perform limited Monte Carlo tests comparing ARMA and FFGN models for different values of CV, ρ, G, and H. Nine (9) sets of parameters were used in these comparisons.

The procedure used was as follows: FFGN traces were generated with population values of H and ρ. ARMA traces were then generated with ϕ and θ interpolated from Table 2.2, and were chosen such that equivalent population values of ρ and large sample expectations of H results. The comparison of H required the use of large sample expectations since a population value of H does not exist for the ARMA model. The behavior of E(H) typically shows two phases. For small n, it either increases (large H) or decreases (small H) with n, then ultimately decays to 0.5 for extremely large n. The initial phase corresponds to reduction of bias with increasing sample size, whereas the secondary decay corresponds to the ultimate influence of the Brownian property of all ARMA models. The value of H used here was the value of $\overline{H}$ reported in Table 2.2, which is the Monte Carlo estimate of the expectation of H for sequences of length 9000. In most cases, this sample size appears to approximately correspond to the region separating small and large sample behavior of K, the sample estimate of H.

In the cases where $G \neq 0$, the skewed marginal distribution was modeled using a three parameter log normal distribution. In these cases, values of ϕ_y and θ_y were obtained from Appendix B. The parameter combinations used in this chapter are given in the first four columns of Table 5.1.

Storage probability curves were constructed following the procedures outlined in Chapter 4. Summary statistics of the same features of simulated flow (number of critical periods, length of most severe and longest critical period, and refill time following the most severe and longest critical period) that were discussed in Chapter 4 are given in Tables 5.1-5.5. Properties of the storage-probability plots are discussed in Section 5.1 below. Section 5.2 gives a method for initializing ARMA(1,1) traces, while critical period information is analyzed in Section 5.3. Finally, Section 5.4 summarizes the findings of this operational comparison of ARMA and FFGN flow generators.

5.2 Computational Considerations

In creating an empirical probability-storage diagram, it is necessary to generate a large number (in this work, 1000) of synthetic streamflow traces, each trace yielding one value of storage for inclusion in the diagram. One possible way to do this is to create one very long synthetic trace, then to use segments of this long trace (say of length 40) as the individual traces. This technique has the disadvantage that the traces are highly dependent which may result in a poor estimate of the storage distribution. A conceptually superior method is to generate independent traces by initializing each trace with a random number. In the case of Lag one Markov generation, if the process variance is σ_x^2, the apparent initial value of the sequence X may be established by simply drawing an initial value of X from a random number generator with the appropriate mean and normal marginal probability distribution and with variance σ_x^2. The subsequent values are then derived using the definition of the Lag one Markov generating process:

$$X_{k+1} = \rho(X_k - \mu) + \varepsilon_{k+1} + \mu \tag{5.1}$$

This initialization procedure is valid for FFGN as well, since the FFGN generator consists of the sum of a number of Lag one Markov processes, each of which may be initialized using this method.

In the case of an ARMA(1,1) generator, some care must be taken in initializing the sequences. The naive approach is to follow the method used for Markov sequences, i.e., to initialize with an independent random number with appropriate mean and normal marginal probability distribution and with initial variance $\sigma_o^2 = \sigma_x^2$. However, if this procedure is used, i.e.,

$$X_{k+1} = \phi X_k + \varepsilon_{k+1} - \theta\varepsilon_k \qquad k = o \tag{5.2}$$

the process variance becomes (assuming a zero mean process)

$$\mathrm{Var}(X_{k+1}) = E[X_{k+1}X_{k+1}] = \phi^2 E[X_k^2] + E[\varepsilon_{k+1}^2] + \theta^2 E[\varepsilon_k^2] \tag{5.3}$$

Now, if the initializing variance is denoted σ_o^2 and the random component ε of eq. 5.2 has variance σ_ε^2, then

$$\mathrm{Var}(X_{k+1}) = \phi^2\sigma_o^2 + \sigma_\varepsilon^2 + \theta^2\sigma_\varepsilon^2 \tag{5.4}$$

Also, assuming the validity of eq. 2.25 (which is of course correct if the initialization is proper),

$$\mathrm{Var}(X_{k+1}) = \phi^2\sigma^2 + \frac{(1+\theta^2)(1-\theta^2)}{1+\theta^2-2\phi\theta}\,\sigma_o^2 \neq \sigma_\varepsilon^2 \tag{5.5}$$

The difficulty with this approach is that it assumes that X_k and ε_k are uncorrelated, which is of course not the case. In fact, it may be easily shown, using the definition of the covariance of X_k and ε_k, that

$$\mathrm{Cov}(X_1,\varepsilon_k) = E(X_{k+1}\varepsilon_{k+1}) = E(\varepsilon_{k+1}^2) = \sigma_\varepsilon^2 \tag{5.6}$$

or

$$\rho_{x\varepsilon} = \frac{\sigma_\varepsilon}{\sigma_x} \tag{5.7}$$

Incorporation of the proper correlation in the initializing results in

$$X_o = \frac{1}{\rho_{x\varepsilon}} [\rho_{x\varepsilon}\varepsilon_o + \sqrt{1-\rho_{x\varepsilon}^2}\, \nu_o] \tag{5.8}$$

where ε_o and ν_o are independent normally distributed random variates both having mean zero and variance σ_ε^2; subsequent values of the process are generated as

$$X_{k+1} = \phi X_k + \varepsilon_{k+1} - \theta\varepsilon_k \qquad k \geq 1 \tag{5.10}$$

The proper variance of X is now preserved for all $k \geq 1$.

X_o is normally distributed with variance $\sigma_o^2 = \sigma_x^2$ and following eq. 2.25

$$\sigma_\varepsilon^2 = \sigma_o^2 \left(\frac{1 - \phi^2}{1 + \theta^2 - 2\phi\theta}\right) \tag{5.11}$$

A straight forward calculation will demonstrate that

$\mathrm{Var}(X_k),\ k \geq 1 = \sigma_x^2$ as desired.

Use of the proper initialization scheme becomes increasingly important as H (hence also ϕ) increases. For small values of ϕ, an alternate initialization procedure is to use a "warmup period" to allow initialization effects to die out. However, this method will be ineffective when ϕ becomes larger than about .8, and in any case the procedure outlined here is superior in that no warmup is required. It should be noted that the effect of using the incorrect initialization procedure is that the initial variance will be too high, which has the effect (especially when ϕ is large) of producing traces with sample means which vary too greatly from the process mean. The result is that spuriously high storage requirements are suggested. This problem will not be encountered if the initialization method given above is employed.

5.3 Comparison of Storage Distributions

In each of Figures 5.1-5.9 the storage distributions corresponding to $D^* = D/\mu_x = 0.9$, 0.7, and 0.5 are given for both ARMA and FFGN models. FFGN results are shown as solid lines, ARMA results are shown as broken lines. In all cases the theoretical distribution describing the numerically generated data is given. Where theoretical results poorly matched the experimental data comparably bad fits for both generators resulted. In these instances (which are noted on the relevent figures) qualitative comparisons are used. Figures 5.1-5.9 are arranged in increasing sequential order of CV, ρ, G, and H.

Generally the storage-probability curve resulting from ARMA generation very nearly coincided with the FFGN curve with notable exceptions shown in Figures 5.2 and 5.9. In the other seven cases that were examined, distinct differences in storage-probability distribution between the two models are not obvious; the discrepancy between curves in these cases may be accounted for by estimation error and error in interpolating values of ϕ and θ from Table 2.2. The FFGN sequences were generated using population values of ρ and H; the ARMA sequences used interpolated values of θ and ϕ which were assumed to give population values of ρ and "psuedo population" values of H. The results shown by O'Connell (1974) concerning bias in H from the ARMA model point out the difficulty in choosing appropriate model parameters. The discrepancies in Figure 5.3 resulted from operational difficulties. Because of restrictions on the feasible range of parameters for the two models (see Figures 2.2 and 2.3) it was only possible to model $H = 0.65$, $\rho = 0.15$ via FFGN while the equivalent model used $\bar{H} = 0.65$ and $\rho = 0.10$ for ARMA generation. Some of the differences in the curves of Figure 5.3 may be attributed to the differences in ρ.

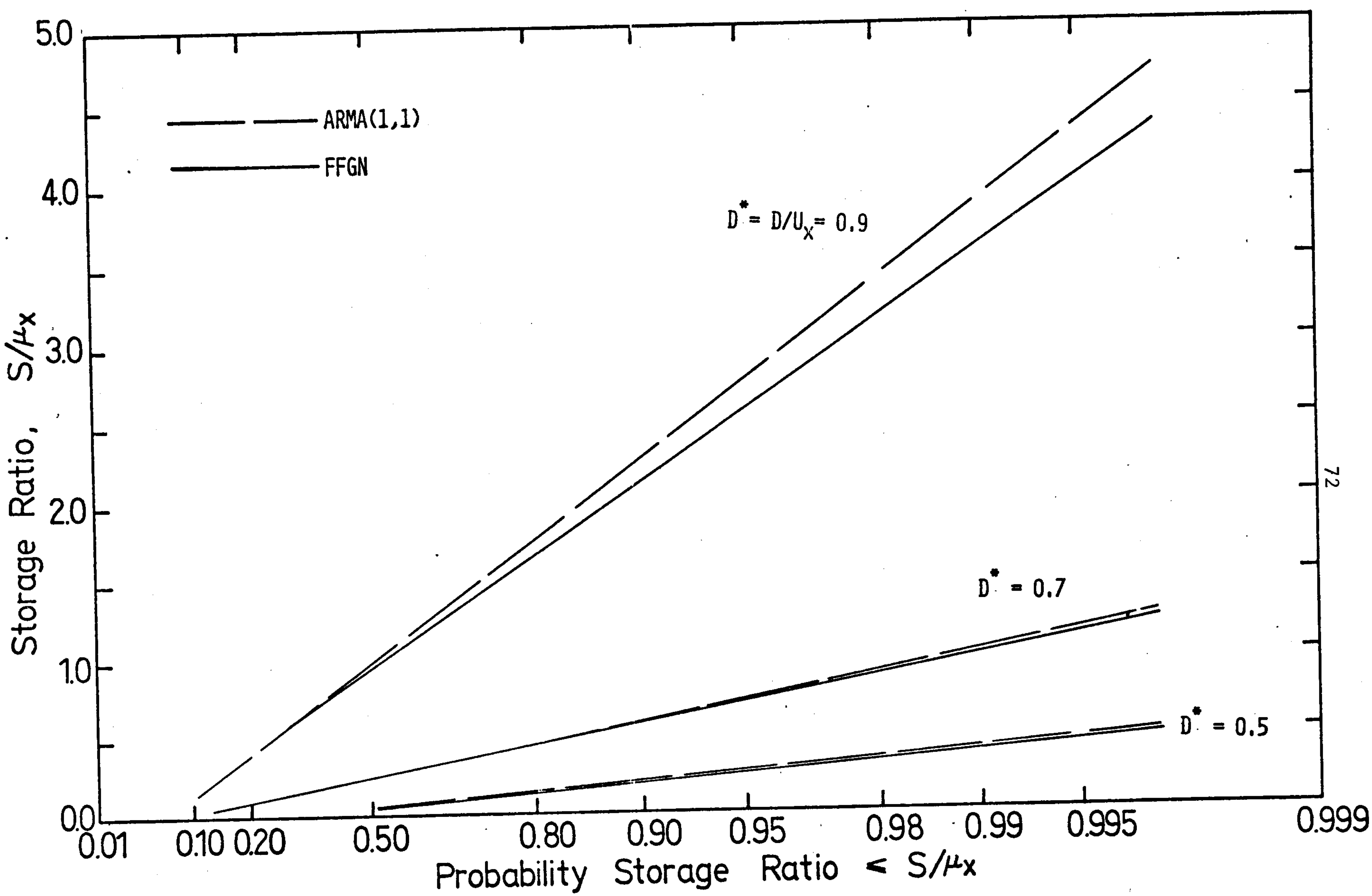


Figure 5.1. Storage Probability Relationships for ARMA and FFGN Sequences: CV = 0.25, ρ = 0.2, G = 0, H = 0.7, N = 40.

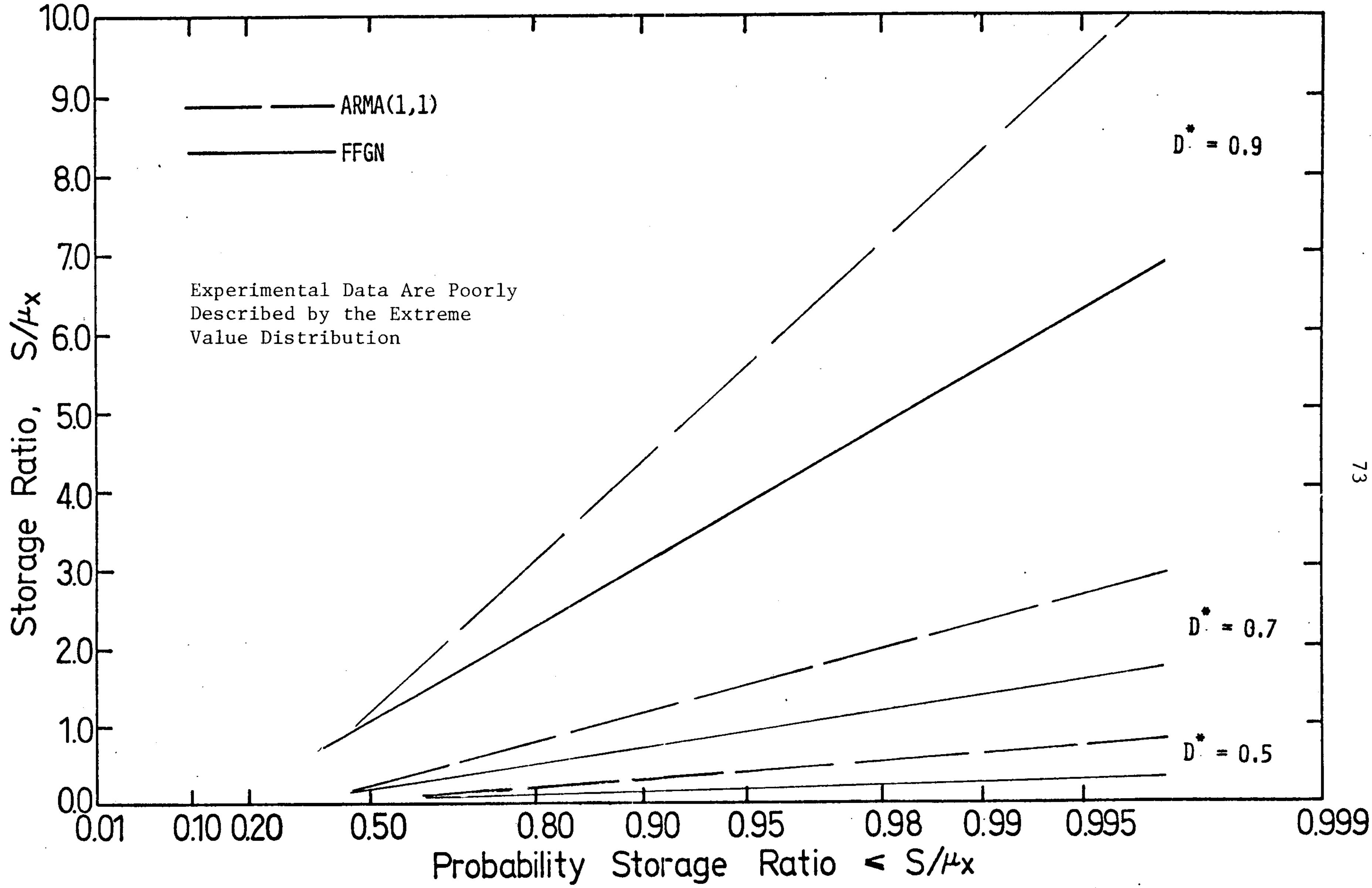


Figure 5.2. Storage Probability Relationships for ARMA and FFGN Sequences: CV = 0.25, ρ = 0.45 G = 0, H = 0.85, N = 40.

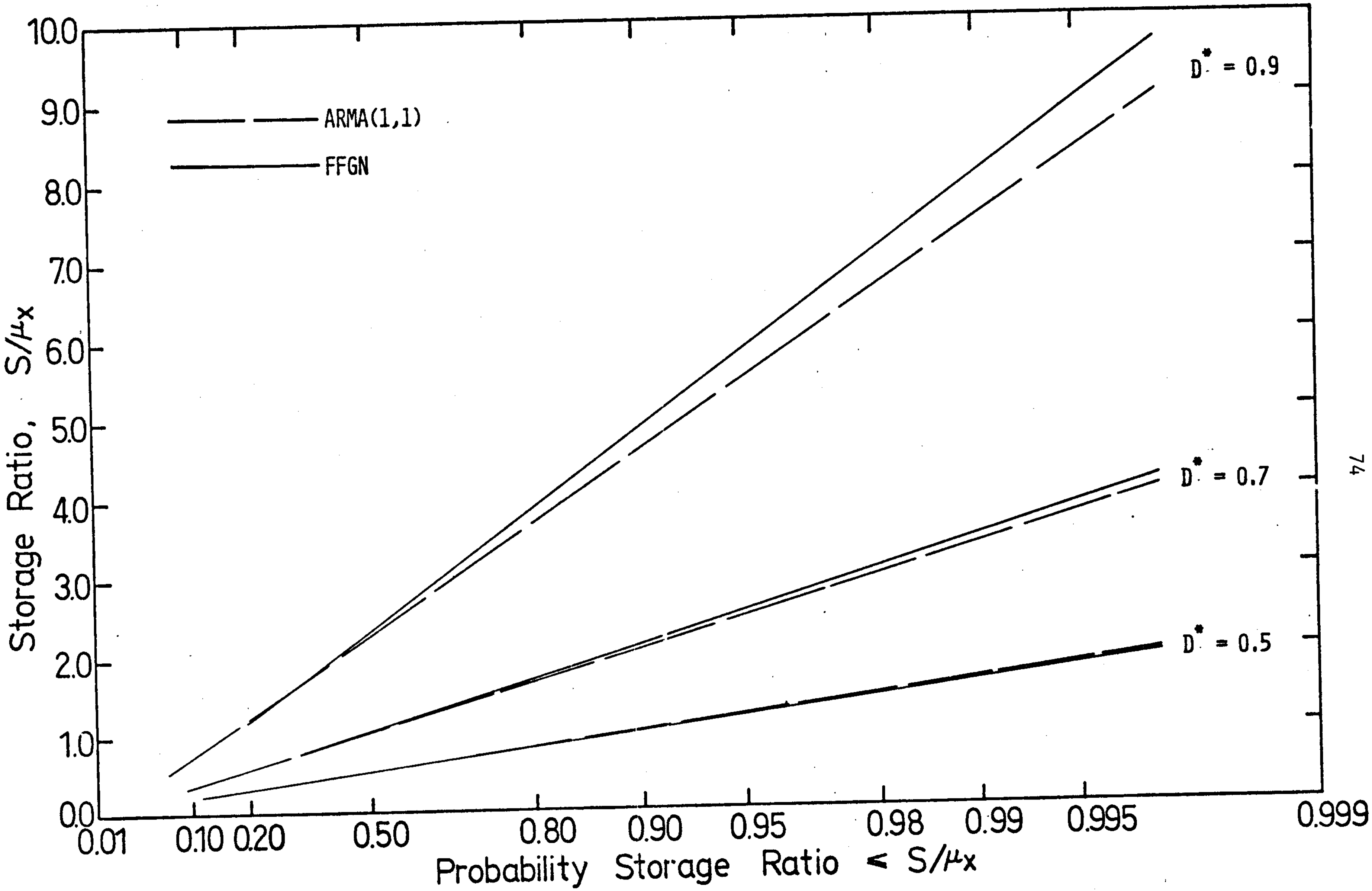


Figure 5.3. Storage Probability Relationships for ARMA and FFGN Sequences: CV = 0.5, G = 0, H = 0.65, ρ = 0.15 (FFGN), ρ = 0.10 (ARMA), N = 40.

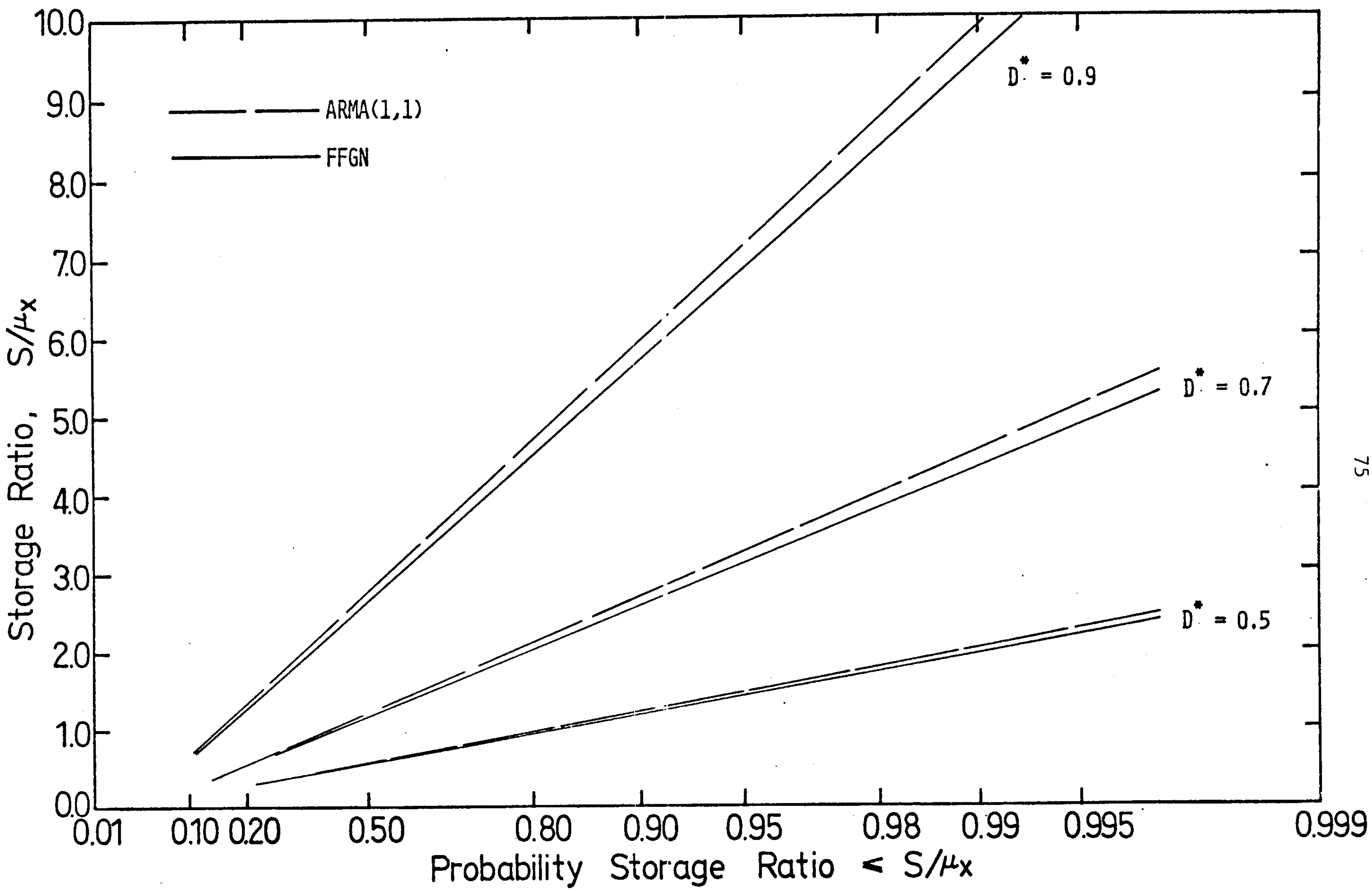


Figure 5.4. Storage Probability Relationships for ARMA and FFGN Sequences: CV = 0.5, ρ = 0.2, G = 0, H = 0.7, N = 40.

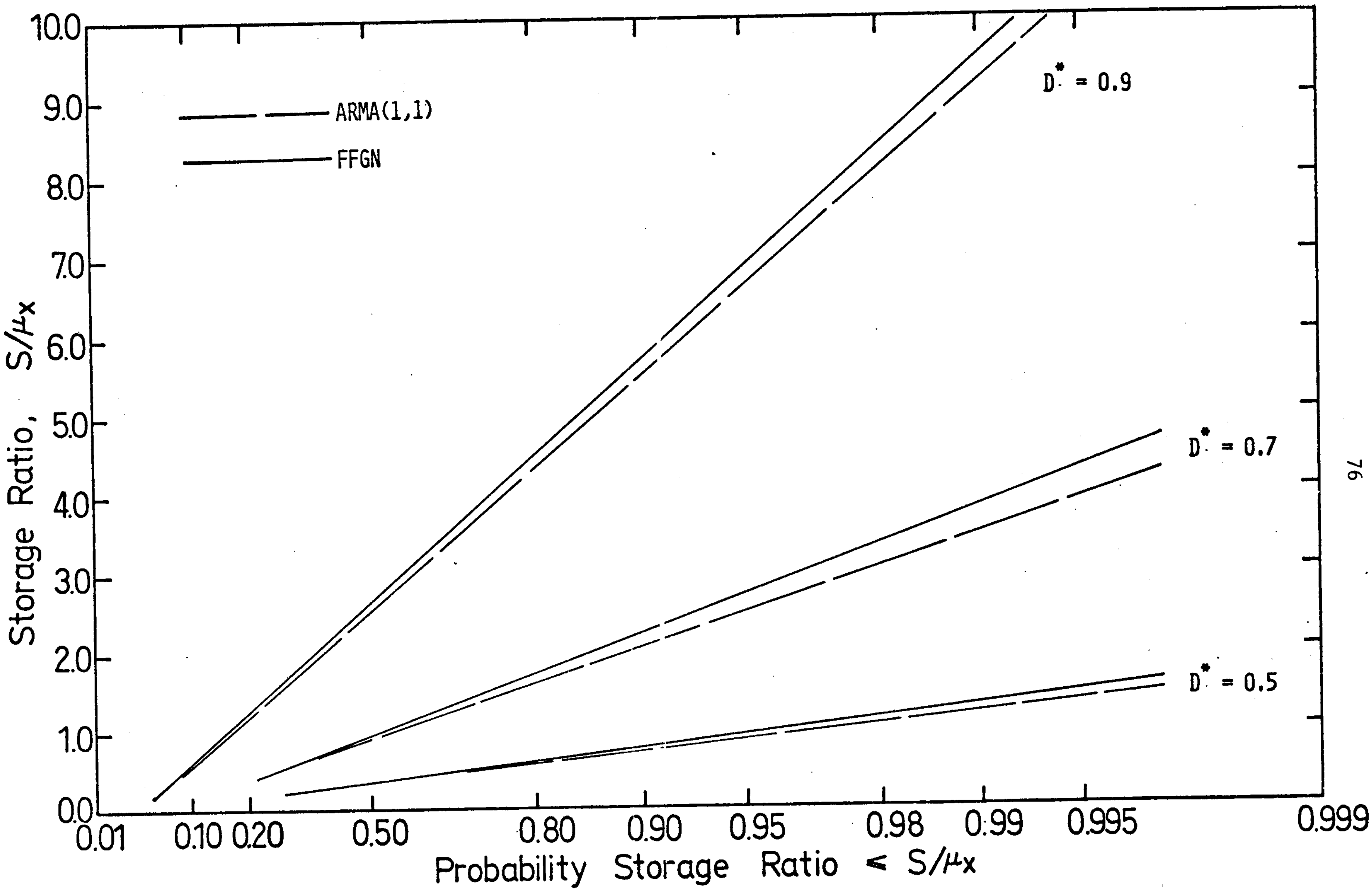


Figure 5.5. Storage Probability Relationships for ARMA and FFGN Sequences: CV = 0.5, ρ = 0.2, G = 1.0, H = 0.7, N = 40.

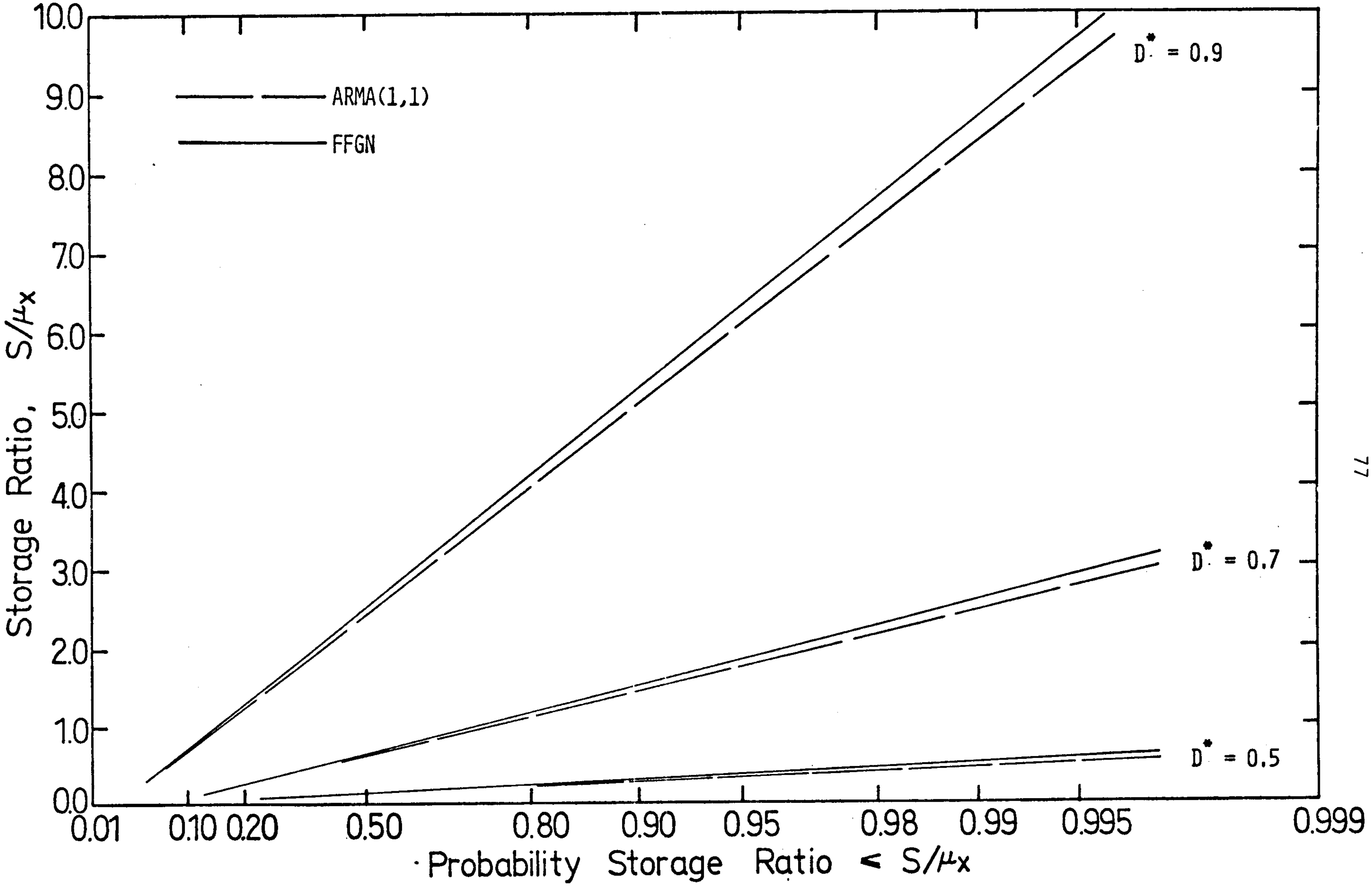


Figure 5.6. Storage Probability Relationships for ARMA and FFGN Sequences: CV = 0.5, ρ = 0.2, G = 2.0, H = 0.7, N = 40.

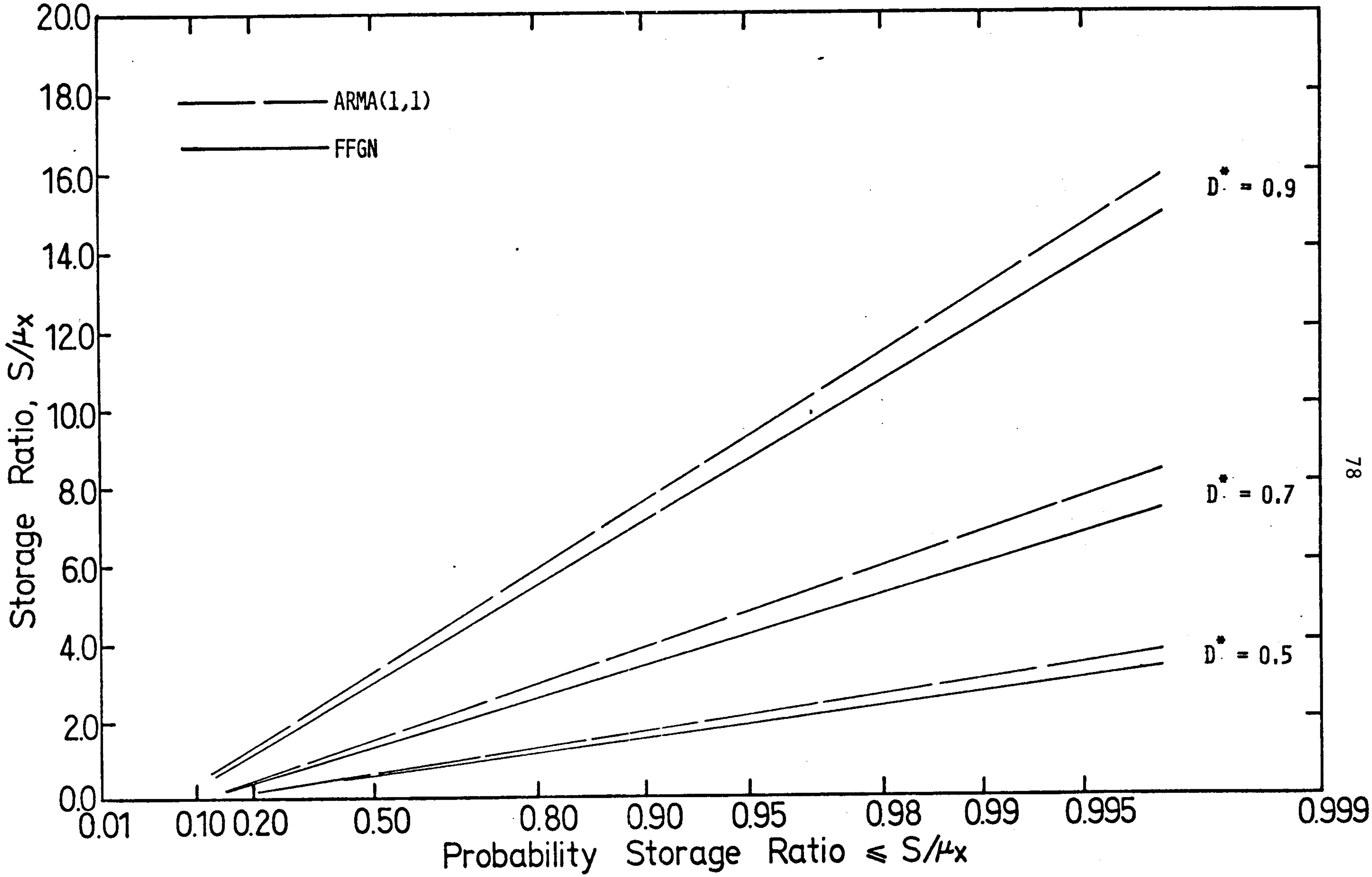


Figure 5.7. Storage Probability Relationships for ARMA and FFGN Sequences: CV = 0.5, $\rho = 0.4$, G = 0, H = 0.75, N = 40.

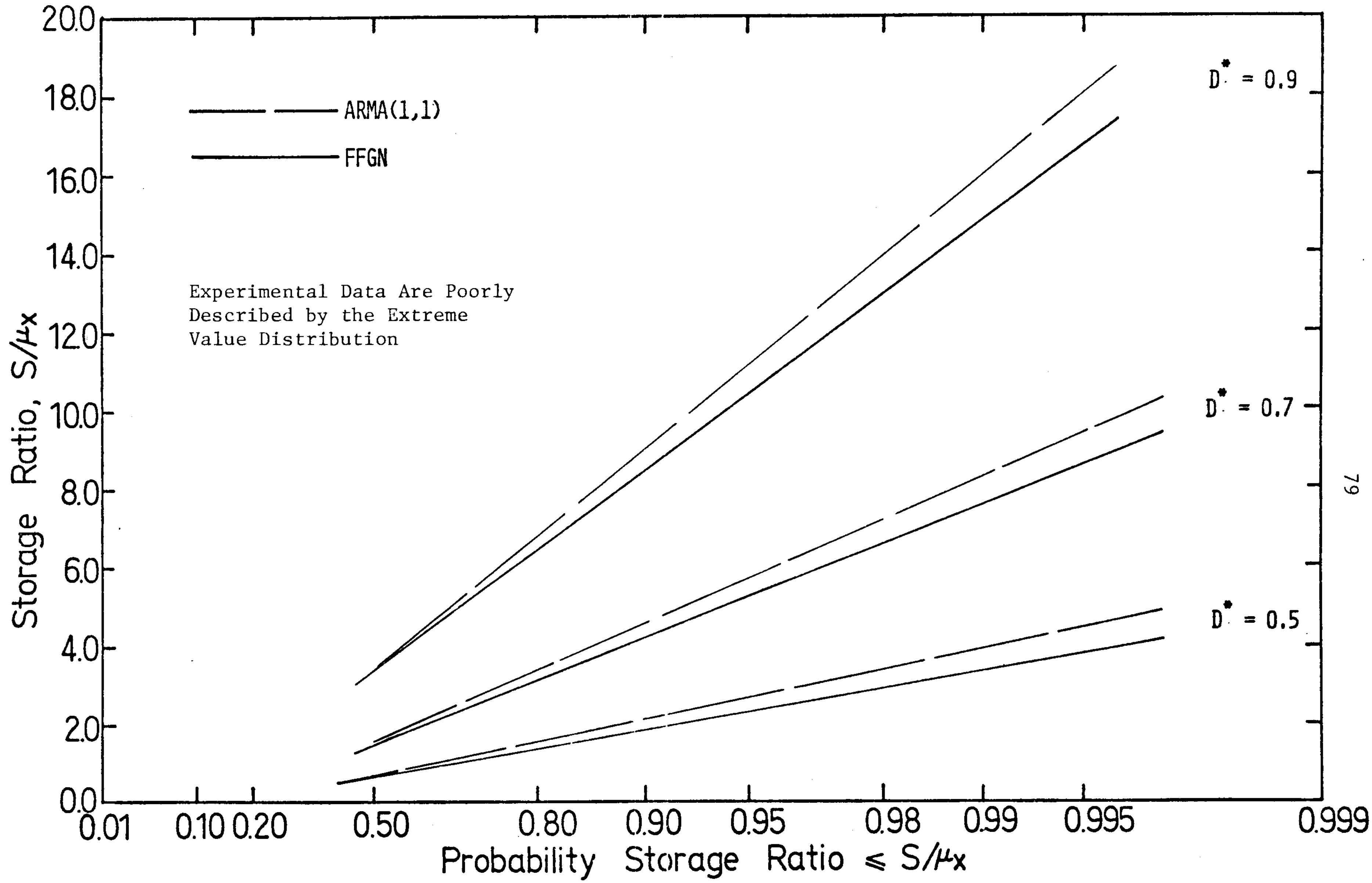


Figure 5.8. Storage Probability Relationships for ARMA and FFGN Sequences: CV = 0.5, ρ = 0.4, G = 0, H = 0.8, N = 40.

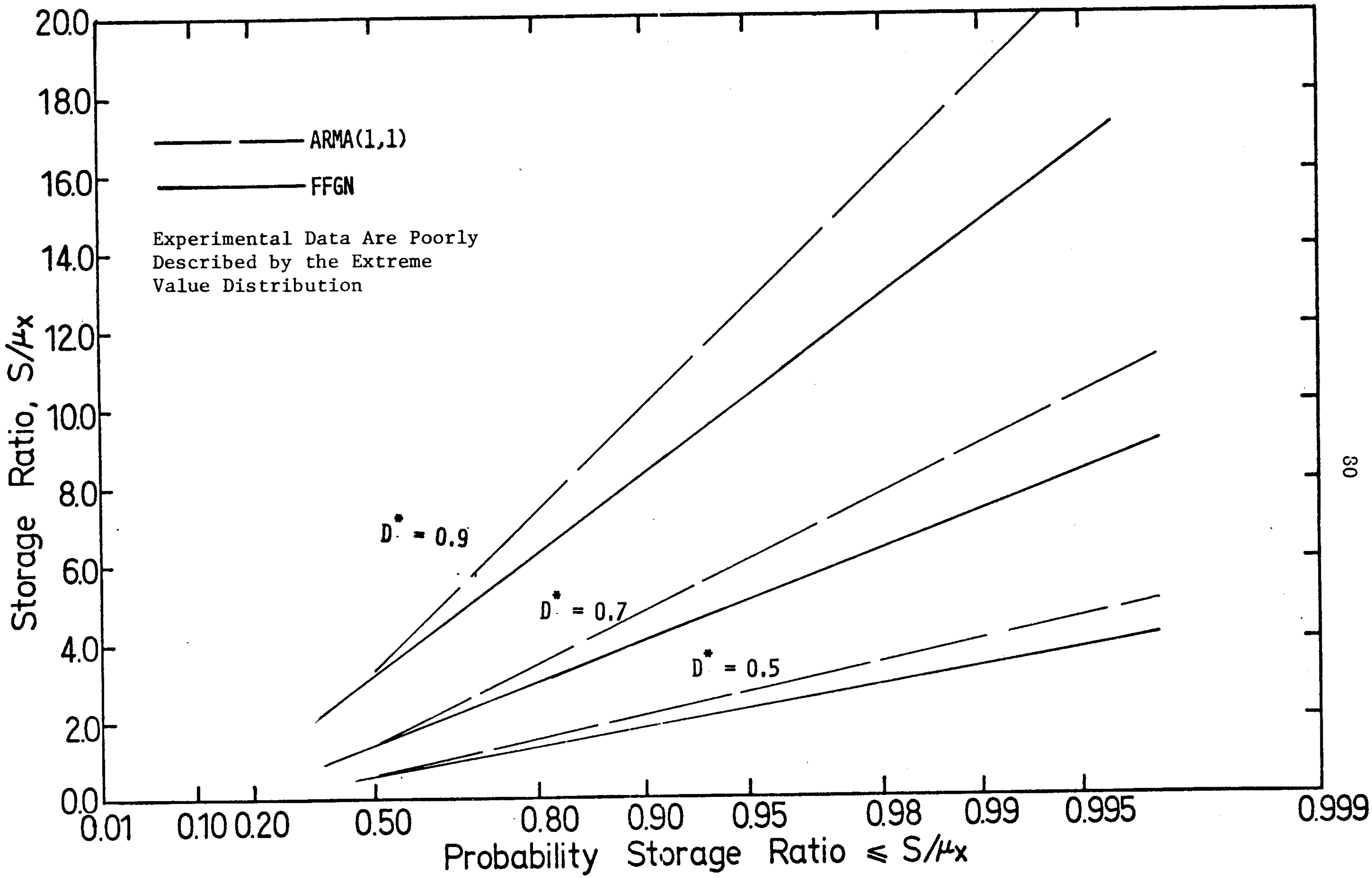


Figure 5.9. Storage Probability Relationships for ARMA and FFGN Sequences: CV = 0.5, $\rho = 0.45$, G = 0, H = 0.85, N = 40.

Generally for H in the vicinity of 0.7 ARMA and FFGN models yield indistinguishable results (Figures 5.3-5.6) irrespective of the values of the other parameters. Figure 5.7 with H = 0.75 shows some possible model differences. It is still possible, however, that most of the differences could be ascribed to estimation and interpolation problems.

Figures 5.2 and 5.9 (H = .85), show areas of substantially different model behavior. In neither of these cases did the extreme value distribution adequately describe the generated storages. The straight lines shown do, however, adequately represent the form of the distributions for purposes of comparison. Figures 5.2 and 5.9 show that for H = .85, and CV = .20 and .50, respectively, the results of the two models diverge significantly. Other test runs covered this range of CV without showing significantly different results between models. Thus, it appears that modeling of sequences with $H \sim .85$ needs to be checked carefully in the future to see which (if either) model is correct.

5.4 Comparison of Critical Period Information

Tables 5.1-5.5 contain the same form of summary information given in Tables 4.2-4.6. To facilitate comparisons between the two generation schemes both ARMA and FFGN critical period information is given in each of Tables 5.1-5.5. Tables 5.1 and 5.3 summarize lengths of most severe and longest critical periods respectively; Tables 5.2 and 5.4 give the corresponding reservoir refill time information. Table 5.5 summarizes the number of critical periods per trace for each of the 9 different sets of flow generator parameters used. Each summary statistic resulted from 1000 independent observations.

5.4.1 Length of the Most Severe Critical Period:

Tables 5.1 and 5.3 show very small differences between the lengths of the most severe and the longest critical periods per trace respectively.

Discussion will therefore be limited to Table 5.1. Consider first the two cases where the storage distributions differed markedly, viz: Figures 5.2 and 5.9 corresponding to parameters listed in rows a and c in Table 5.1. In rows a and c the ARMA models gave rise to critical periods which were longer on the average by about 1 and 2 years respectively and exhibited more variability between traces than was the case with FFGN generated sequences at $D^* = 0.9$; proportionate differences occurred at lower demand levels. While differences between the two models were observed in Figure 5.8 the corresponding critical period length summary information (row b) does not exhibit substantial differences. The distribution of the length of the most severe critical period does, however, exhibit slightly greater variance for the ARMA case which partially explains the larger ARMA storage needs in Figure 5.8. In the remaining situations the differences between the models are minimal except for row d.

Row d corresponds to Figure 5.5 which shows a reasonably good agreement between models. It was quite surprising, therefore, to find longer and more variable severe critical period lengths for the FFGN sequences than for the ARMA sequences. Results for sequences with skew varying and all other parameters held constant agree quite closely for both ARMA and FFGN models. It appears then that the critical period statistics are not always a good index to differences in the storage-probability distributions.

5.4.2 Refill Time Following a Severe Critical Period

Tables 5.2 and 5.4 summarize the number of periods needed for a reservoir to fill following the most severe and the longest critical periods per trace, respectively. There are small differences between the two tables so only Table 5.2 is discussed. The same comment concerning truncation of the refill time that was made in Section 4.4.3b is relevant here, hence, the summary statistics are expected to understate actual refill times.

Similar observations to those made in Section 5.4.1 concerning rows a and c of Tables 5.2 and 5.4 can be made. It appears that with $H \sim 0.85$ the ARMA and FFGN models behave quite differently with the ARMA model overemphasizing the importance of long-term persistence. Unanticipated differences are again obvious in row d.

5.4.3 Number of Critical Periods

Table 5.5 summarizes the number of critical periods (NP) to be expected in any 40 year Trace for three different demand levels. This table shows that differences in NP are not as noticeable between the two models as were differences in critical period lengths and refill times (Table 5.1 and 5.2). Row d does, however, show substantially different model behavior. The most remarkable differences occur at $D^* = 0.5$ and $D^* = 0.7$. This might suggest that the models represent the Noah effect in substantially different ways for this particular parameter combination. Apart from this observation it appears that both models give rise to about the same average value for NP but that the variance of NP is usually larger for ARMA generated sequences. These differences might well be of greater importance in an economic analysis as opposed to the physical analysis carried out here.

5.5 Summary

The objective of this chapter was to compare ARMA and FFGN generation schemes on the basis of storage-probability distribution and critical period statistics. It appears that for $H \lesssim 0.8$ the two models yield essentially the same cumulative probability distribution of storage needed to satisfy a specified demand sequence. For the cases where almost identical distributions resulted there were some small differences in the distributions of numbers of critical periods, length of most severe critical period, and refill time following the most severe critical period.

Table 5.1. Summary Statistics from 1000 Traces for the Length of the Most Severe Critical Period per 40 Year Long Trace for ARMA and FFGN Models.

GENERATOR PARAMETERS				LENGTH OF MOST SEVERE CRITICAL PERIOD DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
				ARMA								
.25	.20	.00	.70	8.98	8.26	1.46	1.74	1.65	4.57	.60	.61	.73
a.25	.45	.00	.85	12.50	13.07	.93	3.32	6.24	3.50	.71	2.07	9.66
.50	.10	.00	.65	11.51	8.65	1.00	4.28	3.82	2.15	1.99	1.51	2.92
.50	.20	.00	.70	13.32	9.75	.86	5.50	5.11	1.89	2.50	2.32	2.92
.50	.40	.00	.75	14.42	10.09	.72	7.03	6.81	1.72	3.27	3.64	2.87
b.50	.40	.00	.80	14.73	12.09	.71	7.45	8.60	1.77	3.24	4.59	3.33
c.50	.45	.00	.85	15.79	14.02	.48	8.42	10.49	1.54	3.39	5.74	3.51
d.50	.20	1.00	.70	14.15	10.32	.83	5.61	5.64	2.13	1.90	1.63	3.05
.50	.20	2.00	.70	15.91	10.38	.63	5.35	4.93	2.39	1.39	1.11	3.69
				FFGN								
.25	.20	.00	.70	8.66	7.83	1.44	1.77	1.42	2.44	.56	.63	1.13
a.25	.45	.00	.85	10.56	10.37	1.27	2.10	3.00	3.74	.42	.76	4.56
.50	.15	.00	.65	11.74	9.39	1.14	4.36	4.18	2.81	2.05	1.58	3.55
.50	.20	.00	.70	13.15	10.11	.93	5.25	5.05	2.49	2.41	2.25	3.74
.50	.40	.00	.75	13.63	10.44	.90	6.25	6.40	2.23	2.93	3.01	3.00
b.50	.40	.00	.80	15.16	11.84	.66	7.60	8.09	1.77	3.40	4.16	3.59
c.50	.45	.00	.85	14.55	11.86	.75	7.04	7.87	1.92	3.22	4.27	3.32
d.50	.20	1.00	.70	16.43	11.70	.56	7.19	7.38	2.01	2.64	3.02	3.62
.50	.20	2.00	.70	16.31	10.66	.63	5.73	5.10	2.09	1.54	1.21	2.19

Table 5.2. Summary Statistics from 1000 Traces for Reservoir Refill Time Following the Most Severe Critical Period per 40 Year Long Trace for ARMA and FFGN Models.

GENERATOR PARAMETERS				REFILL TIME AFTER MOST SEVERE CRITICAL PERIOD								
				DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
				ARMA								
.25	.20	.00	.70	6.66	5.48	1.49	1.90	1.56	2.47	.63	.66	1.00
a .25	.45	.00	.85	5.31	5.82	1.75	2.63	4.14	3.00	.67	1.58	6.99
.50	.10	.00	.65	8.75	6.51	1.13	3.94	3.33	2.32	1.92	1.37	2.59
.50	.20	.00	.70	8.67	6.58	1.00	4.87	4.33	1.97	2.18	1.68	2.62
.50	.40	.00	.75	8.60	6.44	.94	5.72	4.94	1.73	2.74	2.51	2.83
b .50	.40	.00	.80	7.13	6.43	1.37	5.16	5.33	1.83	2.52	2.95	2.94
c .50	.45	.00	.85	5.81	6.09	1.54	4.88	5.50	1.80	2.76	3.78	2.93
d .50	.20	1.00	.70	7.75	6.33	1.19	4.19	3.99	2.41	1.73	1.34	2.60
.50	.20	2.00	.70	7.50	5.95	1.23	3.54	3.52	2.58	1.22	.76	2.40
				FFGN								
.25	.20	.00	.70	7.10	6.00	1.62	1.86	1.44	2.99	.58	.68	1.19
a.25	.45	.00	.85	6.53	6.14	1.58	2.04	2.50	3.26	.42	.74	3.31
.50	.15	.00	.65	8.57	6.70	1.12	4.18	3.57	2.22	1.92	1.23	2.03
.50	.20	.00	.70	7.92	6.14	1.31	4.66	4.00	2.29	2.17	1.50	1.96
.50	.40	.00	.75	8.55	7.04	1.18	5.33	4.72	1.94	2.61	2.45	3.12
b.50	.40	.00	.80	7.56	6.38	1.19	5.57	5.30	1.82	2.92	3.05	3.05
c.50	.45	.00	.85	7.34	6.60	1.27	5.24	5.38	1.98	2.80	3.21	2.84
d.50	.20	1.00	.70	7.09	6.21	1.16	4.99	4.87	2.04	2.12	2.24	3.98
.50	.20	2.00	.70	7.75	6.38	1.14	3.79	3.64	2.58	1.32	.97	3.42

Table 5.3. Summary Statistics from 1000 Traces for the Length of the Longest Critical Period per 40 Year Long Trace for ARMA and FFGN Models.

GENERATOR PARAMETERS				LENGTH OF LONGEST CRITICAL PERIOD								
				DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
						ARMA						
.25	.20	.00	.70	9.36	8.05	1.50	1.91	1.68	4.20	.61	.63	.83
.25	.45	.00	.85	12.68	12.96	.93	3.43	6.24	3.46	.74	2.09	9.29
.50	.10	.00	.65	11.95	8.37	1.04	4.68	3.68	2.17	2.17	1.50	2.76
.50	.20	.00	.70	13.68	9.48	.91	5.86	4.96	1.94	2.69	2.30	2.83
.50	.40	.00	.75	14.59	9.96	.73	7.28	6.70	1.73	3.41	3.61	2.86
.50	.40	.00	.80	14.93	11.93	.72	7.67	8.51	1.78	3.40	4.57	3.29
.50	.45	.00	.85	16.00	13.84	.49	8.64	10.40	1.53	3.55	5.73	3.46
.50	.20	1.00	.70	14.42	10.10	.86	6.00	5.49	2.15	2.10	1.66	2.74
.50	.20	2.00	.70	16.19	10.16	.65	5.76	4.81	2.40	1.48	1.14	3.29
						FFGN						
.25	.20	.00	.70	8.99	7.65	1.47	1.93	1.45	2.17	.56	.65	1.12
.25	.45	.00	.85	10.79	10.25	1.27	2.20	3.00	3.67	.43	.78	4.35
.50	.15	.00	.65	12.17	9.10	1.18	4.71	4.07	2.85	2.26	1.59	3.26
.50	.20	.00	.70	13.50	9.87	.95	5.61	4.97	2.47	2.63	2.23	3.64
.50	.40	.00	.75	13.88	10.25	.93	6.57	6.32	2.22	3.12	3.01	2.90
.50	.40	.00	.80	15.33	11.70	.68	7.87	7.99	1.76	3.55	4.16	3.53
.50	.45	.00	.85	14.73	11.73	.76	7.19	7.81	1.93	3.36	4.24	3.29
.50	.20	1.00	.70	16.61	11.54	.58	7.49	7.31	1.99	2.76	3.01	3.57
.50	.20	2.00	.70	16.58	10.44	.66	6.08	5.03	2.06	1.65	1.25	1.91

Table 5.4. Summary Statistics from 1000 Traces for Reservoir Refill Time Following the Longest Critical Period per 40 Year Long Trace for ARMA and FFGN Models.

GENERATOR PARAMETERS				REFILL TIME AFTER LONGEST CRITICAL PERIOD								
				DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
				ARMA								
.25	.20	.00	.70	6.29	5.50	1.48	1.71	1.45	2.80	.62	.66	1.22
.25	.45	.00	.85	5.13	5.80	1.78	2.52	4.10	3.12	.63	1.54	7.54
.50	.10	.00	.65	8.30	6.51	1.11	3.66	3.38	2.36	1.75	1.29	2.79
.50	.20	.00	.70	8.44	6.67	1.00	4.55	4.37	2.02	1.97	1.63	2.89
.50	.40	.00	.75	8.43	6.50	.95	5.41	4.97	1.74	2.60	2.52	2.90
.50	.40	.00	.80	6.87	6.38	1.36	4.88	5.31	1.89	2.66	2.98	2.83
.50	.45	.00	.85	5.55	6.10	1.59	4.67	5.52	1.86	2.54	3.65	3.05
.50	.20	1.00	.70	7.52	6.35	1.20	3.83	3.96	2.58	1.58	1.24	3.01
.50	.20	2.00	.70	7.21	5.78	1.12	3.35	3.45	2.57	1.17	.73	2.77
				FFGN								
.25	.20	.00	.70	6.62	5.87	1.64	1.68	1.38	3.51	.56	.64	1.13
.25	.45	.00	.85	6.18	6.09	1.62	1.93	2.49	3.43	.42	.74	3.49
.50	.15	.00	.65	7.98	6.67	1.17	3.81	3.53	2.22	1.77	1.23	2.48
.50	.20	.00	.70	7.59	6.18	1.33	4.32	3.85	2.03	2.00	1.49	2.19
.50	.40	.00	.75	8.12	6.91	1.20	5.13	4.85	1.90	2.45	2.44	3.21
.50	.40	.00	.80	7.31	6.37	1.17	5.32	5.32	1.83	2.76	3.00	3.07
.50	.45	.00	.85	7.07	6.52	1.26	5.09	5.39	1.99	2.66	3.22	2.89
.50	.20	1.00	.70	6.95	6.24	1.19	4.74	4.75	1.97	2.05	2.25	4.02
.50	.20	2.00	.70	7.53	6.39	1.16	3.56	3.58	2.61	1.27	.94	3.79

Table 5.5. Summary Statistics from 1000 Traces for the Number of Critical Periods per 40 Year Long Trace for ARMA and FFGN Models.

GENERATOR PARAMETERS				NUMBER OF CRITICAL PERIODS								
				DEMAND = 0.9μ			DEMAND = 0.7μ			DEMAND = 0.5μ		
CV	ρ	G	H	Mean	S.D.	Skew	Mean	S.D.	Skew	Mean	S.D.	Skew
				ARMA								
.25	.20	.00	.70	4.86	2.01	.11	3.52	1.89	.45	.89	1.08	1.35
a.25	.45	.00	.85	3.15	2.05	.57	2.37	2.13	.93	.79	1.35	2.11
.50	.10	.00	.65	4.75	2.22	.21	5.97	1.80	-.04	4.81	1.83	.09
.50	.20	.00	.70	4.21	2.15	.40	5.29	1.87	.10	4.37	1.90	.29
.50	.40	.00	.75	3.52	1.85	.53	4.05	1.75	.36	3.43	1.71	.33
b.50	.40	.00	.80	3.62	2.04	.53	4.05	1.97	.24	3.51	2.09	.29
c.50	.45	.00	.85	3.38	2.19	.59	3.55	2.17	.42	3.19	2.39	.44
d.50	.20	1.00	.70	4.41	2.33	.33	5.56	1.85	.11	3.99	2.02	.31
.50	.20	2.00	.70	4.32	2.35	.40	5.78	1.92	.03	2.84	1.81	.65
				FFGN								
.25	.20	.00	.70	4.72	1.92	.24	3.28	1.74	.38	.74	.92	1.32
a.25	.45	.00	.85	3.55	1.89	.39	2.39	1.87	.59	.54	.93	2.30
.50	.15	.00	.65	4.65	2.25	.20	5.73	1.75	.00	4.55	1.84	.36
.50	.20	.00	.70	4.14	2.06	.35	4.98	1.76	.15	4.13	1.84	.20
.50	.40	.00	.75	3.66	1.88	.44	4.28	1.77	.30	3.56	1.74	.22
b.50	.40	.00	.80	3.31	1.82	.52	3.74	1.75	.33	3.27	1.76	.30
c.50	.45	.00	.85	3.30	1.78	.49	3.63	1.78	.28	2.94	1.91	.41
d.50	.20	1.00	.70	3.39	1.87	.54	3.93	1.70	.31	2.96	1.84	.50
.50	.20	2.00	.70	3.93	2.15	.49	5.30	1.78	.16	2.68	1.69	.56

The comparisons reported in this chapter were made on the basis of $\overline{H}$, a moderate sample size expectation of H. The inability to use a population value of H in ARMA generation may account for some of the discrepancy in results between the two models. Another possible source of the discrepancy is found in the FFGN approximation of FGN. Table 2 of Mandelbrot (1971) shows that the autocorrelation function of FFGN first begins to show a large discrepancy from that of FGN for H between .8 and .85. The discrepancy at lag 40 for H = .85 is about 10%, as compared to only about 4% at H = .80. Since the autocorrelation at large lags is principally responsible for the Hurst effect, this may explain the difference in the FFGN and ARMA model results observed here. Nevertheless, it is unlikely that a 10% error in the large lag autocorrelation function could explain the differences in storage requirements observed in Figures 5.2 and 5.9.

Based on the results of this chapter, it appears that on a physical basis ARMA models can be adequately substituted for FFGN when typical annual values of ρ, G, and CV are encountered provided that H is less than about 0.8. ARMA generated traces are almost an order of magnitude faster than FFGN to compute. While FFGN generation does not require inordinate amounts of computer time the time savings possible with ARMA generation may well be important when extension to a multisite (or multivariate) problem is required.

In typical computer runs FFGN analysis required approximately 190 central processor (CP) seconds (CDC 6400) while ARMA analyses required about 65 CP seconds. In these computer runs 1000 flow sequences of length 40 were generated, sequent peak storage analyses were performed on each trace for three different demand levels and the resulting storages were analyzed preparatory to obtaining storage probability diagrams. The difference in CP seconds, i.e., 125 CP seconds, represents the incremental time FFGN generation required beyond ARMA generation time for these runs.

6. CONCLUSIONS AND RECOMMENDATIONS

6.1 Limitations of Results

The conclusions of this work must, of course, be conditioned on the assumptions made. It will be recalled that the comparisons of flow generators have been made entirely on a physical basis, in particular, distributions of required storage for each given set of parameters for each model based on the Sequent Peak Algorithm provided the basis for the comparisons. This choice was imposed by computational considerations; use of economic criteria would have required routing each synthetic trace through an allocation model to optimize economic benefits; the distribution of economic benefits then could have provided an economic basis for comparison. Even the simplest economic optimization models, however, are much more expensive to use than the Sequent Peak Algorithm (SPA). To some extent, the physical size required based on the SPA should provide an index to economic benefits, the extent to which this might be done must await further research, however.

It should also be noted that the demand pattern has been assumed constant (although the demand level was varied for different runs), and that the operating life was held fixed at forty years. While the choice of operating life and demand pattern is expected to affect the magnitudes of the storage distribution curves, the qualitative differences are unlikely to be affected for other demand patterns and operating lives not drastically different from those used here.

Finally, all comparisons were based on an annual model, so the results are strictly applicable only to over-year storage problems. The results should, however, give an idea of the relative importance of parameters in a within-year generation problem. Quantitative results for the within-year problem would, of course, require further experimentation.

6.2 Conclusions

Subject to the limitations of the research, the results of Chapters 4 and 5 suggest the following conclusions:

1. Correct modelling of long term persistence (Hurst effect) will not in itself guarantee an accurate estimation of the storage distribution needed to satisfy a specific demand pattern. A careful analysis of the relative importance of parameters of the marginal distribution (CV and G) as opposed to persistence (ρ and H) must be made, as well as between short and long term persistence (ρ and H, respectively). The marginal distribution parameters CV and G are much more important, even in cases where substantial long term persistence is present, than had earlier been thought.

2. The autoregressive moving average (ARMA) model proposed by O'Connell (1974) was found to give essentially identical storage distributions to the FFGN generator for $H \lesssim 0.8$. For larger values of H, the storage distributions diverge rapidly. The reason for this divergence is not known; however, since the FFGN model has received wider use than the ARMA approximation, use of the FFGN model is recommended when $H \gtrsim 0.8$.

3. The constraints on feasible combinations of ρ and H for both the ARMA and FFGN models impose severe limitations on the use of both models. This is especially true in annual generation for parameter combinations where both ρ and H have substantial effects on storage distributions. This problem may be alleviated somewhat since the feasible ranges are different for the ARMA and FFGN models, hence a desired parameter combination may be feasible for one model but infeasible for the other. However, there is a large region in ρ-H space where ρ is relatively small and H moderate to large (a combination frequently encountered in operational situations) where neither model is feasible.

At present an approximate empirical accomodation of this problem is to adjust the parameters of the marginal distribution (CV or G) in such a way as to compensate for this deficiency. This procedure could not, however, be recommended for multisite application.

6.3 Recommendations

1. The method used in this research should be extended to within-year storage problems, possibly through use of the Valencia and Schaake (1973) disaggregation scheme. The utility of the results would thereby be greatly enhanced, as most existing reservoirs are used to provide within-year flow buffering.

2. Extension of either (or both) the FFGN model or the ARMA model to include a greater region in ρ-H space is recommended. A possible approach is to approximate the autocorrelation function of FGN in the same way as the FFGN model, using several ARMA (of any necessary order) components to approximate the low frequency component while leaving a larger portion of the covariance to be taken up by a high frequency term. Alternatively, sampling from the spectrum as proposed by Mejia and Rodriguez (1974) or use of a linear filter approach (Borgman, 1969) might provide synthetic sequences with the required autocorrelation function.

3. Further comparisons of the ARMA and FFGN models are needed to determine the reasons for the divergence of the storage distribution curves for $H \gtrsim 0.8$. In connection with such comparisons, it would be extremely useful to develop an equivalent population H value as a function of ϕ and θ for the ARMA model. While this does not appear possible for the pure ARMA model, use of a process which is the sum of a Lag one Markov process and an ARMA(1,1) process shows great promise in that an equivalent population H value can be derived for such a process.

4. It would be extremely useful to determine the extent to which the physical comparisons used in this work (and frequently in reservoir sizing problems) can provide an index to differences in economic benefits computed using the two models. Given the computational expense of economic operation models, a possible approach might be to use a stratified sampling approach, where the synthetic traces corresponding to several reliability levels on the empirical storage-distribution plot would be routed through an economic model. In this way, an idea of the extent to which the physical comparisons could index economic differences might be given.

REFERENCES

Bates, C. L., D. P. Lettenmaier, and S. J. Burges, "Tables of Parameters for the Three Parameter Log Normal Probability Distribution," Technical Report No. 41, Harris Hydraulics Laboratory, Department of Civil Engineering, University of Washington, Seattle, Washington, 143p, 1974.

Borgman, L., "Ocean Wave Simulation for Engineering Design," Journal of the Waterwaves and Harbors Division, Proceedings, ASCE, Vol. 95, No. WW4, pp 557-583, 1969.

Box, G. E. P., and G. M. Jenkins, Time Series Analysis: Forecasting and Control, Holden-Day, San Francisco, California, 553p, 1970.

Burges, S. J., "Use of Stochastic Hydrology to Determine Storage Requirements for Reservoirs - A Critical Analysis," Report EEP-34, Program in Engineering Economic Planning, Stanford University, Stanford, California, 208p, 1970.

Burges, S. J. and R. K. Linsley, "Some Factors Influencing Required Reservoir Storage," Journal of the Hydraulics Division, ASCE, Proceedings paper 8227, Vol. 97, No. HY7, pp 977-991, 1971.

Burges, S. J., D. P. Lettenmaier and C. L. Bates, "Properties of the Three-Parameter Log Normal Probability Distribution," Water Resources Research, Vol. 11, No. 2, pp 229-235, 1975.

Feller, W., "The Asymptotic Distribution of the Range of Sums of Independent Random Variables," Annals of Mathematical Statistics, Vol. 22, pp 427-432, Sept. 1951.

Fiering, M. B., Streamflow Synthesis, Harvard University Press, 133p, 1967.

Gumbel, E. J., Statistics of Extremes, Columbia University Press, New York, N.Y., 1958.

Hurst, H. E., "Long-term Storage Capacities of Reservoirs," Transactions, ASCE, Vol. 116, p770, 1951.

Hurst, H. E., "Methods of Using Long-term Storage in Reservoirs," Proc. Institution of Civil Engineers, paper 6059, pp 519-590, 1956.

Hurst, H. E., R. P. Black and Y. M. Simaika, Long-Term Storage: An Experimental Study, Constable, London, 145p, 1965.

Jackson, B. B., "The Use of Streamflow Models in Planning," Water Resources Research, Vol. 11, No. 1, pp 54-63, 1975.

Jettmar, R. U., "Study of the Required Degree of Numerical Complexity of Synthetic Data Generators," Ph.D. Thesis, The Catholic University of America, 67p, 1974.

Klemes, V., "The Hurst Phenomenon: A Puzzle?," Water Resources Research, Vol. 10, No. 4, pp 675-688, 1974.

Mandelbrot, B. B., "A Fast Fractional Gaussian Noise Generator," Water Resources Research, Vol. 7, No. 3, pp 543-553, 1971.

Mandelbrot, B. B., and J. R. Wallis, "Computer Experiments with Fractional Gaussian Noises: Part 1: Averages and Variances," Water Resources Research, Vol.5, No.1, pp 228-241, 1969.

Mejia, J. M. and I. Rodriguez-Iturbe, "Correlation Links Between Normal and Log Normal Processes," Water Resources Research, Vol.10, No.4, pp 689-690, 1974a.

Mejia, J. M. and I. Rodriguez-Iturbe, "On the Synthesis of Random Field Sampling From the Spectrum: An Application to the Generation of Hydrologic Spatial Processes," Water Resources Research, Vol.10, No.4, pp 705-711, 1974b.

O'Connell, P. E., "A Simple Stochastic Modelling of Hurst's Law," Proc. of International Symposium on Mathematical Models in Hydrology, Warsaw, pp 327-358, 1971.

O'Connell, P. E., "Stochastic Modelling of Long-term Persistence in Streamflow Sequences," Report 1974-2 Hydrology Section, Department of Civil Engineering, Imperial College, London, 284p, 1974.

Valentia, D. and J. C. Schaake, "Disaggregation Processes in Stochastic Hydrology," Water Resources Research, Vol.9, No.3, pp 580-585, 1973.

Wallis, J. R. and N. C. Matalas, "Small Sample Properties of H and K - Estimators of the Hurst Coefficient h," Water Resources Research, Vol.6, No.6, pp 1583-1594, 1970.

Wallis, J. R. and N. C. Matalas, "Sensitivity of Reservoir Design to the Generating Mechanism of Inflows," Water Resources Research, Vol.8, No.3, pp 634-641, 1972.

APPENDIX A: VALUES OF HURST COEFFICIENT FOR USE IN APPROXIMATING THREE PARAMETER LOG NORMALLY DISTRIBUTED SEQUENCES OF FRACTIONAL GAUSSIAN NOISE

A.1 Description of Table A.1

The values in Table A.1 may be used to determine the required value of Hurst's coefficient in the Y (normal transform) domain which will yield a desired approximation to FGN in the X (untransformed) domain when a three parameter log normal marginal probability distribution is desired for synthetic flow generation. The tabulated relationships were derived using equation 3.6, which equates the correlation coefficients in the theoretical FGN and transformed approximations at lags one and any arbitrary lag k, hence the required inputs include the value of k as well as the desired value of the Hurst coefficient in the real domain (H_x) and the standard deviation of the time series in the normal domain (σ_y). It is suggested that an appropriate value of k will be approximately half the desired length of the synthetic sequences; hence if traces of length 40 are desired a value of $k \approx 20$ might be used.

As an example assume that a three parameter log normal distribution is used to generate sequences of length 40 and the value of H to be preserved is $H_x = 0.75$; the value of σ_y is 0.7. Use of the tables for k = 20, $H_x = 0.75$ and $\sigma_y = 0.7$ indicates that a value of H = 0.7705 should be used to generate FFGN sequences which are exponentiated to yield the desired $H_x = 0.75$.

Table A.1 Values of H for Use in Three Parameter Log Normal Generation of FGN Sequences[a]

K = 4

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5502	.5508	.5518	.5533	.5553	.5579	.5611	.5650	.5697	.5755
.60	.6003	.6013	.6030	.6055	.6087	.6126	.6178	.6239	.6312	.6398
.65	.6504	.6517	.6539	.6569	.6609	.6660	.6721	.6795	.6880	.6980
.70	.7005	.7019	.7043	.7078	.7122	.7178	.7245	.7324	.7415	.7516
.75	.7505	.7520	.7545	.7581	.7627	.7684	.7752	.7830	.7919	.8018
.80	.8005	.8020	.8044	.8079	.8123	.8177	.8241	.8314	.8394	.8483
.85	.8504	.8518	.8540	.8571	.8610	.8657	.8712	.8773	.8840	.8912
.90	.9004	.9014	.9032	.9056	.9086	.9122	.9163	.9206	.9257	.9307
.95	.9502	.9508	.9519	.9533	.9550	.9570	.9593	.9617	.9643	.9669

K = 7

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5502	.5508	.5518	.5532	.5551	.5575	.5606	.5643	.5688	.5743
.60	.6003	.6013	.6029	.6052	.6082	.6120	.6167	.6224	.6292	.6371
.65	.6504	.6516	.6536	.6564	.6602	.6648	.6706	.6774	.6853	.6946
.70	.7004	.7018	.7040	.7072	.7113	.7165	.7227	.7301	.7386	.7483
.75	.7505	.7519	.7542	.7575	.7618	.7671	.7735	.7809	.7893	.7987
.80	.8005	.8018	.8042	.8074	.8116	.8167	.8227	.8296	.8374	.8459
.85	.8504	.8517	.8538	.8567	.8604	.8650	.8702	.8761	.8827	.8897
.90	.9003	.9014	.9030	.9054	.9083	.9118	.9158	.9202	.9250	.9300
.95	.9502	.9508	.9518	.9532	.9549	.9569	.9591	.9616	.9641	.9667

K = 10

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5502	.5508	.5517	.5531	.5550	.5573	.5603	.5639	.5683	.5735
.60	.6003	.6012	.6028	.6050	.6079	.6115	.6160	.6215	.6279	.6355
.65	.6504	.6515	.6534	.6561	.6597	.6642	.6696	.6761	.6837	.6925
.70	.7004	.7017	.7038	.7069	.7108	.7157	.7217	.7287	.7368	.7461
.75	.7504	.7518	.7540	.7572	.7613	.7664	.7724	.7795	.7876	.7968
.80	.8004	.8018	.8040	.8071	.8111	.8160	.8218	.8285	.8360	.8443
.85	.8504	.8516	.8536	.8565	.8601	.8645	.8696	.8754	.8818	.8887
.90	.9003	.9013	.9030	.9052	.9081	.9115	.9155	.9198	.9245	.9295
.95	.9502	.9508	.9518	.9532	.9548	.9568	.9590	.9615	.9640	.9666

[a]Refer to eq. 3.6 for respective notation.

K = 20

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5502	.5507	.5516	.5530	.5547	.5570	.5597	.5631	.5672	.5721
.60	.6003	.6011	.6026	.6046	.6073	.6107	.6146	.6198	.6258	.6327
.65	.6503	.6514	.6531	.6556	.6589	.6630	.6679	.6738	.6808	.6888
.70	.7004	.7015	.7035	.7062	.7098	.7143	.7198	.7262	.7336	.7422
.75	.7504	.7516	.7537	.7566	.7603	.7650	.7705	.7771	.7846	.7931
.80	.8004	.8016	.8037	.8065	.8102	.8146	.8202	.8264	.8335	.8414
.85	.8504	.8515	.8534	.8560	.8594	.8636	.8684	.8739	.8801	.8867
.90	.9003	.9013	.9028	.9050	.9077	.9110	.9148	.9190	.9236	.9285
.95	.9502	.9508	.9518	.9531	.9547	.9567	.9589	.9612	.9637	.9663

K = 40

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5502	.5507	.5516	.5528	.5545	.5566	.5592	.5624	.5663	.5709
.60	.6003	.6010	.6024	.6043	.6068	.6099	.6136	.6184	.6238	.6302
.65	.6503	.6513	.6529	.6552	.6582	.6619	.6664	.6719	.6782	.6856
.70	.7004	.7014	.7032	.7057	.7090	.7131	.7181	.7240	.7308	.7387
.75	.7504	.7515	.7534	.7560	.7594	.7637	.7688	.7749	.7818	.7897
.80	.8004	.8015	.8034	.8060	.8094	.8136	.8187	.8245	.8312	.8387
.85	.8504	.8514	.8532	.8556	.8588	.8627	.8673	.8725	.8784	.8848
.90	.9003	.9012	.9027	.9047	.9074	.9105	.9142	.9183	.9228	.9275
.95	.9502	.9508	.9517	.9530	.9546	.9565	.9587	.9610	.9635	.9661

K = 70

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5502	.5507	.5515	.5527	.5543	.5564	.5589	.5619	.5656	.5699
.60	.6002	.6010	.6023	.6040	.6064	.6094	.6130	.6173	.6224	.6284
.65	.6503	.6512	.6527	.6548	.6576	.6611	.6654	.6705	.6764	.6833
.70	.7003	.7013	.7030	.7053	.7084	.7122	.7169	.7224	.7287	.7361
.75	.7503	.7514	.7531	.7556	.7588	.7628	.7676	.7733	.7798	.7873
.80	.8003	.8014	.8032	.8056	.8088	.8126	.8176	.8231	.8294	.8366
.85	.8503	.8513	.8530	.8553	.8584	.8621	.8664	.8715	.8771	.8834
.90	.9003	.9011	.9026	.9045	.9071	.9101	.9137	.9177	.9221	.9268
.95	.9502	.9507	.9517	.9529	.9545	.9564	.9585	.9608	.9633	.9659

K = 100

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5502	.5506	.5515	.5527	.5542	.5562	.5586	.5616	.5652	.5694
.60	.6002	.6010	.6022	.6039	.6062	.6090	.6125	.6167	.6216	.6274
.65	.6503	.6511	.6526	.6546	.6573	.6607	.6648	.6696	.6753	.6819
.70	.7003	.7013	.7029	.7051	.7081	.7117	.7 62	.7214	.7275	.7346
.75	.7503	.7513	.7530	.7554	.7585	.7623	.7669	.7723	.7786	.7858
.80	.8003	.8013	.8030	.8054	.8085	.8123	.8169	.8222	.8284	.8353
.85	.8503	.8513	.8529	.8552	.8581	.8617	.8659	.8708	.8763	.8824
.90	.9003	.9011	.9025	.9044	.9069	.9099	.9134	.9173	.9216	.9263
.95	.9502	.9507	.9516	.9529	.9544	.9563	.9584	.9607	.9632	.9657

Table A.1 (Cont.)

K = 400

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5501	.5506	.5514	.5524	.5539	.5557	.5579	.5606	.5637	.5675
.60	.6002	.6008	.6019	.6034	.6055	.6080	.6110	.6147	.6189	.6239
.65	.6502	.6510	.6522	.6540	.6563	.6592	.6628	.6669	.6718	.6775
.70	.7003	.7011	.7024	.7044	.7069	.7101	.7139	.7184	.7236	.7296
.75	.7503	.7511	.7526	.7546	.7572	.7605	.7645	.7692	.7746	.7806
.80	.8003	.8012	.8026	.8047	.8074	.8107	.8146	.8193	.8247	.8308
.85	.8503	.8511	.8525	.8545	.8571	.8603	.8641	.8685	.8735	.8791
.90	.9003	.9010	.9023	.9040	.9063	.9090	.9122	.9159	.9200	.9244
.95	.9502	.9507	.9515	.9527	.9542	.9560	.9580	.9603	.9627	.9652

K = 700

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5501	.5506	.5513	.5524	.5537	.5555	.5576	.5602	.5632	.5668
.60	.6002	.6008	.6018	.6033	.6052	.6076	.6105	.6140	.6180	.6228
.65	.6502	.6509	.6521	.6538	.6560	.6588	.6621	.6660	.6707	.6760
.70	.7003	.7010	.7023	.7041	.7065	.7095	.7131	.7173	.7223	.7280
.75	.7503	.7511	.7524	.7543	.7568	.7599	.7637	.7681	.7732	.7791
.80	.8003	.8011	.8025	.8044	.8070	.8101	.8139	.8183	.8234	.8292
.85	.8503	.8511	.8524	.8543	.8568	.8598	.8634	.8676	.8724	.8778
.90	.9002	.9010	.9022	.9039	.9060	.9087	.9118	.9153	.9193	.9237
.95	.9502	.9507	.9515	.9527	.9541	.9559	.9579	.9601	.9625	.9650

K = 1000

SIGMA Y

HX	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.55	.5501	.5506	.5513	.5523	.5537	.5554	.5574	.5600	.5629	.5665
.60	.6002	.6008	.6018	.6032	.6051	.6074	.6102	.6136	.6175	.6221
.65	.6502	.6509	.6521	.6537	.6558	.6585	.6617	.6655	.6700	.6751
.70	.7002	.7010	.7022	.7040	.7063	.7092	.7126	.7167	.7215	.7270
.75	.7503	.7510	.7523	.7542	.7566	.7596	.7632	.7675	.7724	.7781
.80	.8003	.8011	.8024	.8043	.8067	.8098	.8134	.8177	.8226	.8263
.85	.8503	.8510	.8523	.8542	.8566	.8595	.8630	.8671	.8718	.8771
.90	.9002	.9009	.9021	.9038	.9059	.9085	.9115	.9150	.9189	.9232
.95	.9502	.9507	.9515	.9526	.9541	.9558	.9578	.9600	.9623	.9648

APPENDIX B: VALUES OF THE PARAMETERS OF AN ARMA (1,1) MODEL FOR USE IN APPROXIMATING THREE PARAMETER LOG NORMALLY DISTRIBUTED ARMA (1,1) SEQUENCES

B.1 Description of Tables B.1 and B.2

Tables B.1 and B.2 give the required values of the parameters ϕ and θ of an ARMA (1,1) model in the Y (normal) domain to approximate three parameter log normally distributed ARMA (1,1) sequences in the X (real) domain. The transformation was effected using equations 3.13 and 3.14 which equate the correlation coefficients in a theoretical ARMA (1,1) model with those of its transformed approximation at lags one and two.

Tables B.1a and B.1b give the required values of ϕ_y and θ_y, respectively as a function of σ_y, the standard deviation of the sequences in the Y (normal) domain and the desired parameter values ϕ_x and θ_x in the X (real) domain. The same information is given in Tables B.2 for a smaller range of ϕ_x ($.905 \leq \phi_x \leq .995$) in which ARMA (1,1) models usually provide the best modeling of the Hurst effect (O'Connell, 1971). As discussed in Chapter 3, some combinations of ϕ_x, θ_x, and σ_y result in imaginary values of θ_y; such values are denoted by an asterik (*) in the tables.

Table B.1a Values of ϕ_y for Given ϕ_x, θ_x, σ_y

SIGMA Y = .10

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1501	.2502	.3503	.4505	.5506	.6507	.7507	.8505	.9502
.150	.0500	.1500	.2501	.3502	.4504	.5505	.6506	.7506	.8505	.9502
.250	.0500	.1499	.2500	.3501	.4503	.5504	.6505	.7506	.8505	.9502
.350	.0499	.1499	.2499	.3500	.4501	.5503	.6504	.7505	.8504	.9502
.450	.0499	.1498	.2498	.3499	.4500	.5501	.6503	.7504	.8504	.9502
.550	.0499	.1498	.2498	.3498	.4499	.5500	.6501	.7502	.8503	.9502
.650	.0499	.1498	.2497	.3497	.4498	.5499	.6500	.7501	.8502	.9501
.750	.0499	.1497	.2497	.3497	.4497	.5498	.6499	.7500	.8501	.9501
.850	.0499	.1497	.2497	.3496	.4497	.5497	.6498	.7499	.8500	.9500
.950	.0499	.1497	.2496	.3496	.4497	.5497	.6496	.7499	.8500	.9500

SIGMA Y = .20

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1503	.2508	.3514	.4520	.5526	.6528	.7527	.8521	.9509
.150	.0499	.1500	.2504	.3510	.4516	.5521	.6525	.7525	.8520	.9509
.250	.0498	.1497	.2500	.3505	.4511	.5516	.6521	.7522	.8519	.9509
.350	.0497	.1495	.2496	.3500	.4505	.5511	.6516	.7519	.8517	.9506
.450	.0497	.1493	.2493	.3496	.4500	.5505	.6511	.7514	.8514	.9506
.550	.0496	.1492	.2490	.3492	.4495	.5500	.6505	.7509	.8511	.9507
.650	.0496	.1490	.2488	.3489	.4491	.5495	.6500	.7504	.8507	.9506
.750	.0496	.1490	.2487	.3487	.4489	.5492	.6496	.7500	.8503	.9504
.850	.0495	.1489	.2486	.3485	.4487	.5490	.6493	.7497	.8500	.9502
.950	.0495	.1489	.2486	.3485	.4486	.5489	.6492	.7495	.8498	.9500

SIGMA Y = .30

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1506	.2518	.3532	.4547	.5558	.6564	.7561	.8547	.9520
.150	.0498	.1500	.2509	.3522	.4536	.5549	.6557	.7557	.8545	.9519
.250	.0496	.1494	.2500	.3511	.4524	.5538	.6547	.7550	.8542	.9519
.350	.0494	.1489	.2491	.3500	.4512	.5525	.6536	.7542	.8538	.9518
.450	.0492	.1484	.2484	.3490	.4500	.5512	.6524	.7532	.8533	.9517
.550	.0491	.1480	.2478	.3481	.4489	.5500	.6512	.7521	.8525	.9516
.650	.0490	.1478	.2473	.3474	.4480	.5489	.6500	.7510	.8517	.9513
.750	.0490	.1476	.2469	.3469	.4474	.5481	.6491	.7500	.8508	.9509
.850	.0489	.1475	.2467	.3466	.4470	.5476	.6484	.7493	.8500	.9504
.950	.0489	.1474	.2467	.3465	.4468	.5474	.6481	.7489	.8496	.9500

Table B.1a (Cont.)

SIGMA Y = .40

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1511	.2532	.3559	.4585	.5605	.6615	.7609	.8584	.9534
.150	.0496	.1500	.2516	.3540	.4566	.5588	.6602	.7601	.8580	.9534
.250	.0492	.1489	.2500	.3520	.4545	.5566	.6586	.7590	.8575	.9533
.350	.0489	.1479	.2484	.3500	.4522	.5546	.6566	.7576	.8568	.9532
.450	.0486	.1471	.2470	.3481	.4500	.5523	.6544	.7559	.8558	.9530
.550	.0484	.1464	.2458	.3465	.4480	.5500	.6521	.7539	.8546	.9528
.650	.0482	.1459	.2449	.3452	.4463	.5480	.6500	.7518	.8530	.9523
.750	.0481	.1455	.2443	.3443	.4451	.5465	.6483	.7500	.8514	.9516
.850	.0480	.1453	.2439	.3437	.4443	.5456	.6471	.7487	.8500	.9507
.950	.0480	.1452	.2436	.3435	.4440	.5451	.6465	.7480	.8493	.9500

SIGMA Y = .50

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1518	.2553	.3595	.4636	.5667	.6681	.7670	.8629	.9553
.150	.0493	.1500	.2527	.3565	.4606	.5641	.6661	.7658	.8624	.9552
.250	.0487	.1482	.2500	.3533	.4572	.5610	.6636	.7642	.8616	.9551
.350	.0481	.1466	.2474	.3500	.4536	.5575	.6606	.7620	.8606	.9549
.450	.0477	.1452	.2451	.3469	.4500	.5537	.6571	.7594	.8592	.9547
.550	.0473	.1440	.2431	.3442	.4467	.5500	.6535	.7563	.8572	.9543
.650	.0470	.1431	.2416	.3420	.4439	.5468	.6500	.7530	.8548	.9536
.750	.0469	.1426	.2406	.3405	.4419	.5443	.6471	.7500	.8522	.9526
.850	.0468	.1422	.2400	.3396	.4406	.5426	.6452	.7478	.8500	.9512
.950	.0467	.1421	.2397	.3391	.4400	.5419	.6443	.7467	.8488	.9500

SIGMA Y = .60

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1527	.2580	.3643	.4702	.5746	.6764	.7745	.8683	.9574
.150	.0490	.1500	.2541	.3598	.4659	.5709	.6736	.7728	.8676	.9573
.250	.0480	.1473	.2500	.3550	.4609	.5664	.6701	.7706	.8666	.9571
.350	.0471	.1447	.2460	.3500	.4555	.5612	.6657	.7676	.8652	.9569
.450	.0464	.1425	.2424	.3452	.4500	.5556	.6607	.7638	.8633	.9566
.550	.0458	.1407	.2394	.3410	.4449	.5500	.6552	.7593	.8606	.9561
.650	.0454	.1394	.2370	.3377	.4406	.5450	.6500	.7545	.8572	.9552
.750	.0451	.1385	.2354	.3352	.4374	.5411	.6456	.7500	.8533	.9536
.850	.0450	.1379	.2344	.3338	.4354	.5386	.6426	.7466	.8500	.9517
.950	.0449	.1377	.2339	.3331	.4345	.5374	.6411	.7449	.8481	.9500

SIGMA Y = .70

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1540	.2615	.3704	.4786	.5843	.6662	.7832	.8745	.9597
.150	.0485	.1500	.2559	.3641	.4726	.5793	.6826	.7811	.8736	.9596
.250	.0471	.1460	.2500	.3572	.4656	.5732	.6779	.7782	.8724	.9594
.350	.0456	.1422	.2442	.3500	.4579	.5660	.6721	.7743	.8706	.9592
.450	.0447	.1369	.2388	.3430	.4500	.5580	.6652	.7693	.8681	.9588
.550	.0438	.1362	.2342	.3367	.4425	.5500	.6576	.7632	.8646	.9582
.650	.0432	.1342	.2306	.3316	.4361	.5427	.6500	.7565	.8601	.9571
.750	.0427	.1327	.2281	.3279	.4312	.5368	.6435	.7500	.8548	.9552
.850	.0425	.1319	.2265	.3256	.4281	.5329	.6390	.7450	.8500	.9525
.950	.0424	.1315	.2259	.3246	.4267	.5311	.6368	.7425	.8472	.9500

Table B.1a (Cont.)

SIGMA Y = .80

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1556	.2660	.3781	.4889	.5900	.6977	.7930	.8812	.9622
.150	.0479	.1500	.2583	.3696	.4810	.5896	.6935	.7905	.8802	.9621
.250	.0456	.1443	.2500	.3601	.4716	.5816	.6874	.7870	.8788	.9619
.350	.0439	.1388	.2417	.3500	.4611	.5720	.6799	.7822	.8767	.9616
.450	.0423	.1340	.2338	.3400	.4500	.5612	.6706	.7759	.8737	.9612
.550	.0410	.1299	.2270	.3308	.4392	.5500	.6605	.7681	.8694	.9604
.650	.0400	.1268	.2216	.3232	.4298	.5395	.6500	.7590	.8636	.9591
.750	.0394	.1246	.2178	.3176	.4225	.5306	.6407	.7500	.8567	.9569
.850	.0390	.1233	.2154	.3141	.4178	.5250	.6340	.7428	.8500	.9534
.950	.0388	.1228	.2144	.3125	.4157	.5223	.6307	.7391	.8460	.9500

SIGMA Y = .90

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1577	.2719	.3877	.5013	.6097	.7109	.8039	.8885	.9648
.150	.0470	.1500	.2614	.3766	.4913	.6018	.7056	.8010	.8874	.9647
.250	.0441	.1420	.2500	.3639	.4792	.5918	.6985	.7969	.8858	.9645
.350	.0413	.1342	.2383	.3500	.4652	.5796	.6893	.7913	.8834	.9642
.450	.0389	.1270	.2270	.3358	.4500	.5653	.6778	.7838	.8800	.9637
.550	.0370	.1209	.2169	.3225	.4348	.5500	.6644	.7740	.8749	.9629
.650	.0355	.1161	.2087	.3112	.4210	.5351	.6500	.7622	.8679	.9614
.750	.0344	.1128	.2027	.3026	.4101	.5224	.6367	.7500	.8590	.9588
.850	.0338	.1107	.1990	.2971	.4028	.5137	.6269	.7398	.8500	.9545
.950	.0335	.1098	.1974	.2946	.3995	.5094	.6219	.7343	.8443	.9500

SIGMA Y = 1.00

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.1605	.2793	.3996	.5162	.6254	.7254	.8155	.8960	.9674
.150	.0458	.1500	.2655	.3855	.5039	.6162	.7194	.8124	.8948	.9673
.250	.0416	.1386	.2500	.3688	.4887	.6041	.7113	.8079	.8931	.9671
.350	.0376	.1275	.2336	.3500	.4705	.5869	.7004	.8016	.8906	.9668
.450	.0339	.1169	.2172	.3301	.4500	.5706	.6864	.7928	.8868	.9663
.550	.0307	.1074	.2020	.3106	.4286	.5500	.6692	.7811	.8812	.9654
.650	.0282	.0997	.1891	.2933	.4082	.5269	.6500	.7663	.8729	.9639
.750	.0263	.0940	.1793	.2797	.3914	.5102	.6312	.7500	.8619	.9610
.850	.0252	.0904	.1731	.2707	.3799	.4966	.6166	.7356	.8500	.9559
.950	.0247	.0887	.1702	.2665	.3744	.4699	.6069	.7274	.8420	.9500

Table B.1b Values of θ_y for Given ϕ_x, θ_x, σ_y

SIGMA Y = .10

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0496	.0494	.0493	.0492	.0493	.0495	.0495	.0497	.0499
.150	.1505	.1500	.1496	.1494	.1493	.1493	.1494	.1495	.1497	.1499
.250	.2513	.2505	.2500	.2497	.2495	.2494	.2494	.2495	.2497	.2499
.350	.3523	.3512	.3505	.3500	.3497	.3495	.3495	.3495	.3496	.3499
.450	.4537	.4522	.4512	.4505	.4500	.4497	.4496	.4495	.4496	.4498
.550	.5560	.5536	.5523	.5512	.5505	.5500	.5497	.5496	.5496	.5496
.650	.6598	.6564	.6541	.6524	.6513	.6505	.6500	.6497	.6496	.6497
.750	.7673	.7615	.7575	.7547	.7527	.7514	.7505	.7500	.7497	.7497
.850	.8888	.8749	.8660	.8601	.8562	.8535	.8516	.8506	.8500	.8498
.950	*	*	*	*	.9849	.9660	.9579	.9534	.9510	.9500

SIGMA Y = .20

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0484	.0475	.0470	.0469	.0470	.0474	.0480	.0487	.0495
.150	.1522	.1500	.1486	.1478	.1474	.1474	.1476	.1481	.1487	.1495
.250	.2551	.2521	.2500	.2487	.2480	.2477	.2477	.2481	.2487	.2495
.350	.3593	.3549	.3520	.3500	.3488	.3481	.3479	.3481	.3486	.3494
.450	.4655	.4592	.4549	.4519	.4500	.4488	.4482	.4481	.4485	.4493
.550	.5753	.5659	.5594	.5549	.5520	.5500	.5488	.5483	.548[illegible]	.5491
.650	.6929	.6774	.6670	.6600	.6552	.6520	.6500	.6488	.6484	.6489
.750	.8357	.8023	.7825	.7697	.7614	.7558	.7522	.7500	.7489	.7486
.850	*	*	.9442	.8988	.8775	.8650	.8573	.8526	.8500	.8490
.950	*	*	*	*	*	*	*	.9661	.9543	.9500

SIGMA Y = .30

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0464	.0443	.0432	.0430	.0433	.0442	.0455	.0472	.0490
.150	.1550	.1500	.1468	.1449	.1441	.1441	.1447	.1458	.1472	.1490
.250	.2619	.2547	.2500	.2470	.2454	.2448	.2450	.2458	.2471	.2469
.350	.3718	.3615	.3545	.3500	.3472	.3458	.3453	.3457	.3469	.3488
.450	.4869	.4716	.4613	.4544	.4500	.4473	.4460	.4458	.4466	.4485
.550	.6125	.5881	.5721	.5615	.5545	.5500	.5474	.5462	.5463	.5481
.650	.7673	.7194	.6914	.6737	.6622	.6547	.6500	.6474	.6465	.6477
.750	*	.9333	.8382	.7995	.7773	.7637	.7551	.7500	.7474	.7474
.850	*	*	*	*	.9317	.8681	.8675	.8561	.8500	.8476
.950	*	*	*	*	*	*	*	*	.9607	.9500

SIGMA Y = .40

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0436	.0397	.0379	.0375	.0383	.0399	.0423	.0451	.0483
.150	.1592	.1500	.1442	.1409	.1395	.1396	.1407	.1426	.1452	.1483
.250	.2720	.2587	.2500	.2447	.2418	.2408	.2412	.2426	.2450	.2482
.350	.3910	.3712	.3583	.3500	.3450	.3424	.3417	.3425	.3446	.3479
.450	.5217	.4906	.4710	.4581	.4500	.4452	.4429	.4426	.4440	.4474
.550	.6610	.6250	.5921	.5714	.5582	.5500	.5453	.5432	.5436	.5467
.650	*	.8067	.7338	.6956	.6728	.6586	.6500	.6453	.6437	.6459
.750	*	*	*	.8590	.8042	.7759	.7594	.7500	.7454	.7454
.850	*	*	*	*	*	.9396	.8844	.8613	.8500	.8460
.950	*	*	*	*	*	*	*	*	.9727	.9500

SIGMA Y = .50

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0397	.0338	.0310	.0306	.0319	.0346	.0383	.0427	.0475
.150	.1650	.1500	.1408	.1357	.1336	.1338	.1357	.1388	.1428	.1475
.250	.2864	.2641	.2500	.2415	.2371	.2357	.2364	.2388	.2425	.2473
.350	.4195	.3851	.3635	.3500	.3421	.3382	.3372	.3385	.3418	.3468
.450	.5786	.5197	.4848	.4633	.4500	.4424	.4389	.4385	.4409	.4461
.550	.8514	.6884	.6226	.5857	.5634	.5500	.5425	.5394	.5401	.5450
.650	*	*	.8165	.7305	.6883	.6641	.6500	.6425	.6402	.6438
.750	*	*	*	*	.8522	.7943	.7655	.7500	.7427	.7429
.850	*	*	*	*	*	*	.9141	.8689	.8500	.8436
.950	*	*	*	*	*	*	*	*	*	.9500

SIGMA Y = .60

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0348	.0263	.0225	.0222	.0245	.0286	.0339	.0400	.0466
.150	.1728	.1500	.1364	.1292	.1265	.1271	.1300	.1345	.1402	.1466
.250	.3064	.2714	.2500	.2376	.2313	.2295	.2308	.2344	.2397	.2463
.350	.4623	.4046	.3704	.3500	.3384	.3329	.3316	.3339	.3386	.3457
.450	.6841	.5639	.5043	.4701	.4500	.43 9	.4340	.4337	.4372	.4446
.550	*	.8302	.6712	.6061	.5704	.5500	.5391	.5348	.5360	.5431
.650	*	*	*	.7911	.7109	.6715	.6500	.6391	.6360	.6413
.750	*	*	*	*	*	.8229	.7739	.7500	.7393	.7399
.850	*	*	*	*	*	*	*	.8798	.8500	.8410
.950	*	*	*	*	*	*	*	*	*	.9500

SIGMA Y = .70

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0287	.0172	.0125	.0126	.0161	.0219	.0291	.0372	.0457
.150	.1830	.1500	.1310	.1213	.1181	.1194	.1237	.1299	.1375	.1457
.250	.3344	.2810	.2500	.2327	.2245	.2224	.2245	.2296	.2368	.2453
.350	.5306	.4321	.3797	.3500	.3339	.3267	.3256	.3268	.3353	.3445
.450	*	.6377	.5322	.4792	.4500	.4346	.4282	.4282	.4333	.4431
.550	*	*	.7606	.6358	.5797	.5500	.5349	.5294	.5314	.5411
.650	*	*	*	*	.7454	.6815	.6500	.6350	.6311	.6386
.750	*	*	*	*	*	.8736	.7854	.7500	.7353	.7365
.850	*	*	*	*	*	*	*	.8960	.8500	.8377
.950	*	*	*	*	*	*	*	*	*	.9500

Table B.1b (Cont.)

SIGMA Y = .80

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0212	.0064	.0009	.0019	.0071	.0150	.0244	.0345	.0449
.150	.1966	.1500	.1245	.1122	.1087	.1110	.1170	.1253	.1348	.1449
.250	.3749	.2937	.2500	.2269	.2165	.2145	.2179	.2247	.2338	.2444
.350	.6624	.4724	.3919	.3500	.3286	.3196	.3188	.3234	.3318	.3433
.450	*	.8078	.5741	.4914	.4500	.4295	.4217	.4222	.4291	.4415
.550	*	*	*	.6823	.5922	.5500	.5300	.5233	.5264	.5390
.650	*	*	*	*	.8049	.6951	.6500	.6301	.6256	.6356
.750	*	*	*	*	*	*	.8017	.7500	.7305	.7328
.850	*	*	*	*	*	*	*	.9229	.8500	.8339
.950	*	*	*	*	*	*	*	*	*	.9500

SIGMA Y = .90

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0121	-.0062	-.0120	-.0096	-.0022	.0081	.0199	.0321	.0441
.150	.2148	.1500	.1166	.1018	.0984	.1023	.1104	.1208	.1323	.1441
.250	.4377	.3109	.2500	.2200	.2076	.2060	.2110	.2199	.2311	.2435
.350	*	.5367	.4085	.3500	.3223	.3117	.3115	.3179	.3285	.3422
.450	*	*	.6440	.5079	.4500	.4236	.4145	.4160	.4250	.4401
.550	*	*	*	.7680	.6095	.5500	.5242	.5166	.5213	.5370
.650	*	*	*	*	*	.7143	.6500	.6244	.6196	.6329
.750	*	*	*	*	*	*	.8259	.7500	.7251	.7289
.850	*	*	*	*	*	*	*	*	.8500	.8295
.950	*	*	*	*	*	*	*	*	*	.9500

SIGMA Y = 1.00

PHI X

THETA X	.050	.150	.250	.350	.450	.550	.650	.750	.850	.950
.050	.0500	.0013	-.0205	-.0260	-.0215	-.0114	.0017	.0158	.0299	.0435
.150	.2400	.1500	.1074	.0901	.0876	.0935	.1040	.1167	.1301	.1435
.250	.5520	.3347	.2500	.2120	.1979	.1973	.2043	.2154	.2266	.2427
.350	*	.6626	.4317	.3500	.3151	.3033	.3042	.3126	.3254	.3412
.450	*	*	.6070	.5313	.4500	.4169	.4068	.4097	.4210	.4387
.550	*	*	*	*	.6343	.5500	.5177	.5095	.5161	.5351
.650	*	*	*	*	*	.7430	.6500	.6180	.6133	.6302
.750	*	*	*	*	*	*	.8669	.7500	.7189	.7250
.850	*	*	*	*	*	*	*	*	.8500	.8247
.950	*	*	*	*	*	*	*	*	*	.9500

Table B.2a Values of θ_y for Given ϕ_x, θ_x, σ_y (fine scale)

SIGMA Y = .10

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0498	.0498	.0498	.0499	.0499	.0499	.0499	.0499	.0500	.0500
.150	.1498	.1498	.1498	.1499	.1499	.1499	.1499	.1499	.1500	.1500
.250	.2498	.2498	.2498	.2498	.2499	.2499	.2499	.2499	.2500	.2500
.350	.3498	.3498	.3498	.3498	.3498	.3499	.3499	.3499	.3500	.3500
.450	.4497	.4497	.4498	.4498	.4498	.4498	.4499	.4499	.4499	.4500
.550	.5497	.5497	.5497	.5497	.5498	.5498	.5498	.5499	.5499	.5500
.650	.6496	.6497	.6497	.6497	.6497	.6498	.6498	.6498	.6499	.6500
.750	.7497	.7497	.7497	.7497	.7497	.7497	.7497	.7498	.7499	.7499
.850	.8498	.8498	.8498	.8498	.8498	.8498	.8498	.8498	.8498	.8499
.950	.9503	.9502	.9502	.9501	.9500	.9500	.9499	.9499	.9499	.9499

SIGMA Y = .20

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0492	.0492	.0493	.0494	.0495	.0496	.0497	.0498	.0499	.0500
.150	.1492	.1493	.1493	.1494	.1495	.1496	.1497	.1498	.1499	.1500
.250	.2491	.2492	.2493	.2494	.2495	.2496	.2497	.2498	.2498	.2499
.350	.3490	.3491	.3492	.3493	.3494	.3495	.3496	.3497	.3498	.3499
.450	.4489	.4489	.4490	.4491	.4493	.4494	.4495	.4496	.4498	.4499
.550	.5487	.5488	.5489	.5490	.5491	.5492	.5494	.5495	.5497	.5499
.650	.6486	.6486	.6487	.6488	.6489	.6490	.6492	.6494	.6496	.6498
.750	.7487	.7487	.7487	.7487	.7488	.7489	.7490	.7492	.7494	.7496
.850	.8493	.8492	.8491	.8491	.8490	.8490	.8490	.8491	.8493	.8496
.950	.9513	.9509	.9506	.9503	.9501	.9499	.9498	.9497	.9496	.9496

SIGMA Y = .30

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0481	.0483	.0485	.0487	.0489	.0491	.0493	.0495	.0497	.0499
.150	.1482	.1484	.1485	.1487	.1489	.1491	.1493	.1495	.1497	.1499
.250	.2481	.2483	.2484	.2486	.2488	.2490	.2492	.2495	.2497	.2499
.350	.3478	.3480	.3482	.3484	.3486	.3489	.3491	.3494	.3496	.3499
.450	.4475	.4477	.4479	.4481	.4484	.4486	.4489	.4492	.4495	.4498
.550	.5471	.5472	.5475	.5477	.5480	.5483	.5486	.5489	.5493	.5498
.650	.6468	.6469	.6471	.6473	.6475	.6478	.6482	.6486	.6491	.6497
.750	.7470	.7470	.7471	.7471	.7473	.7475	.7478	.7482	.7487	.7495
.850	.8483	.8481	.8480	.8479	.8478	.8478	.8478	.8480	.8484	.8492
.950	.9531	.9522	.9514	.9506	.9502	.9498	.9495	.9492	.9491	.9491

Table B.2a (Cont.)

SIGMA Y = .40

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0468	.0471	.0475	.0478	.0481	.0485	.0488	.0491	.0495	.0498
.150	.1469	.1472	.1475	.1478	.1482	.1485	.1488	.1491	.1495	.1498
.250	.2467	.2470	.2473	.2477	.2480	.2483	.2487	.2491	.2494	.2498
.350	.3462	.3466	.3469	.3473	.3477	.3481	.3485	.3489	.3493	.3498
.450	.4456	.4460	.4463	.4467	.4472	.4476	.4481	.4486	.4491	.4497
.550	.5449	.5452	.5456	.5460	.5465	.5470	.5475	.5482	.5488	.5496
.650	.6443	.6446	.6449	.6452	.6457	.6462	.6468	.6475	.6484	.6494
.750	.7446	.7447	.7448	.7449	.7452	.7456	.7461	.7468	.7477	.7491
.850	.8470	.8466	.8464	.8462	.8461	.8461	.8462	.8465	.8471	.8485
.950	.9558	.9541	.9526	.9514	.9504	.9496	.9490	.9486	.9483	.9435

SIGMA Y = .50

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0453	.0458	.0462	.0467	.0472	.0477	.0482	.0487	.0492	.0497
.150	.1453	.1458	.1463	.1468	.1473	.1477	.1482	.1487	.1492	.1497
.250	.2450	.2455	.2460	.2465	.2470	.2475	.2481	.2486	.2492	.2497
.350	.3443	.3449	.3454	.3460	.3465	.3471	.3477	.3484	.3490	.3497
.450	.4433	.4439	.4445	.4451	.4457	.4464	.4471	.4479	.4487	.4496
.550	.5422	.5427	.5433	.5439	.5446	.5454	.5463	.5472	.5483	.5494
.650	.6412	.6417	.6421	.6427	.6434	.6442	.6452	.6463	.6476	.6491
.750	.7416	.7417	.7419	.7422	.7426	.7432	.7440	.7451	.7466	.7486
.850	.8452	.8447	.8443	.8440	.8438	.8438	.8440	.8445	.8456	.8476
.950	.9598	.9568	.9543	.9523	.9507	.9494	.9484	.9477	.9474	.9476

SIGMA Y = .60

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0436	.0442	.0449	.0456	.0463	.0469	.0476	.0483	.0490	.0497
.150	.1437	.1443	.1450	.1456	.1463	.1470	.1476	.1483	.1490	.1497
.250	.2432	.2439	.2446	.2453	.2460	.2467	.2474	.2481	.2489	.2496
.350	.3422	.3430	.3437	.3445	.3453	.3461	.3469	.3478	.3486	.3495
.450	.4408	.4416	.4424	.4432	.4441	.4451	.4461	.4471	.4482	.4494
.550	.5390	.5396	.5407	.5416	.5426	.5437	.5449	.5462	.5476	.5492
.650	.6376	.6382	.6389	.6398	.6408	.6419	.6433	.6448	.6467	.6488
.750	.7380	.7381	.7384	.7388	.7395	.7404	.7415	.7431	.7452	.7481
.850	.8429	.8422	.8416	.8412	.8410	.8411	.8414	.8422	.8437	.8469
.950	.9658	.9606	.9567	.9535	.9510	.9491	.9476	.9466	.9461	.9465

SIGMA Y = .70

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0419	.0427	.0436	.0444	.0453	.0461	.0470	.0479	.0487	.0496
.150	.1420	.1428	.1436	.1445	.1453	.1462	.1470	.1479	.1487	.1496
.250	.2413	.2422	.2431	.2440	.2449	.2458	.2467	.2476	.2486	.2495
.350	.3400	.3410	.3419	.3429	.3439	.3450	.3461	.3472	.3483	.3494
.450	.4380	.4391	.4401	.4413	.4425	.4437	.4450	.4463	.4478	.4492
.550	.5356	.5367	.5378	.5390	.5404	.5418	.5434	.5451	.5470	.5489
.650	.6335	.6344	.6354	.6365	.6379	.6394	.6412	.6433	.6457	.6485
.750	.7336	.7339	.7343	.7350	.7359	.7372	.7388	.7409	.7438	.7476
.850	.8402	.8392	.8384	.8379	.8377	.8378	.8383	.8395	.8416	.8460
.950	.9758	.9663	.9599	.9551	.9515	.9487	.9467	.9453	.9446	.9452

Table B.2a (Cont.)

SIGMA Y = .80

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0402	.0413	.0423	.0433	.0444	.0454	.0464	.0474	.0485	.0495
.150	.1403	.1413	.1423	.1434	.1444	.1454	.1464	.1474	.1485	.1495
.250	.2395	.2406	.2416	.2427	.2438	.2449	.2460	.2472	.2483	.2494
.350	.3378	.3390	.3402	.3414	.3426	.3439	.3452	.3466	.3479	.3493
.450	.4352	.4365	.4379	.4393	.4408	.4423	.4439	.4456	.4473	.4491
.550	.5320	.5334	.5349	.5364	.5381	.5400	.5419	.5440	.5463	.5487
.650	.6291	.6302	.6315	.6331	.6348	.6368	.6391	.6417	.6447	.6481
.750	.7287	.7291	.7298	.7307	.7320	.7337	.7359	.7387	.7423	.7470
.850	.8368	.8356	.8346	.8340	.8338	.8340	.8348	.8364	.8394	.8450
.950	*	.9755	.9644	.9572	.9521	.9482	.9455	.9437	.9428	.9436

SIGMA Y = .90

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0387	.0400	.0412	.0423	.0435	.0447	.0459	.0471	.0483	.0494
.150	.1388	.1400	.1412	.1423	.1435	.1447	.1459	.1471	.1482	.1494
.250	.2378	.2390	.2403	.2416	.2428	.2441	.2454	.2467	.2480	.2493
.350	.3357	.3371	.3385	.3400	.3414	.3429	.3445	.3460	.3476	.3492
.450	.4325	.4341	.4357	.4374	.4392	.4410	.4429	.4448	.4468	.4489
.550	.5284	.5301	.5319	.5338	.5359	.5381	.5405	.5430	.5456	.5485
.650	.6244	.6259	.6276	.6295	.6317	.6342	.6370	.6402	.6437	.6478
.750	.7233	.7239	.7249	.7262	.7279	.7301	.7329	.7363	.7408	.7465
.850	.8329	.8313	.8302	.8295	.8294	.8298	.8310	.8332	.8370	.8440
.950	*	*	.9713	.9601	.9528	.9477	.9441	.9417	.9407	.9421

SIGMA Y = 1.00

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.0375	.0388	.0402	.0415	.0429	.0442	.0455	.0468	.0481	.0494
.150	.1375	.1388	.1402	.1415	.1428	.1441	.1455	.1468	.1481	.1494
.250	.2363	.2377	.2391	.2406	.2420	.2435	.2449	.2464	.2478	.2493
.350	.3338	.3354	.3370	.3387	.3403	.3421	.3436	.3455	.3473	.3491
.450	.4300	.4318	.4337	.4357	.4377	.4398	.4419	.4441	.4464	.4488
.550	.5250	.5270	.5291	.5314	.5338	.5364	.5391	.5420	.5450	.5483
.650	.6197	.6216	.6237	.6261	.6287	.6317	.6350	.6387	.6428	.6475
.750	.7175	.7184	.7197	.7214	.7236	.7264	.7298	.7341	.7393	.7460
.850	.8283	.8264	.8252	.8245	.8245	.8252	.8268	.8297	.8346	.8431
.950	*	*	.9844	.9641	.9537	.9469	.9423	.9394	.9383	.9403

Table B.2b Values of θ_y for Given ϕ_x, θ_x, σ_y (fine scale)

SIGMA Y = .10

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9054	.9154	.9253	.9353	.9452	.9552	.9652	.9751	.9851	.9950
.150	.9054	.9153	.9253	.9353	.9452	.9552	.9652	.9751	.9851	.9950
.250	.9054	.9153	.9253	.9353	.9452	.9552	.9652	.9751	.9851	.9950
.350	.9053	.9153	.9253	.9353	.9452	.9552	.9652	.9751	.9851	.9950
.450	.9053	.9153	.9253	.9352	.9452	.9552	.9651	.9751	.9851	.9950
.550	.9052	.9152	.9252	.9352	.9452	.9552	.9651	.9751	.9851	.9950
.650	.9052	.9152	.9252	.9352	.9452	.9551	.9651	.9751	.9851	.9950
.750	.9051	.9151	.9251	.9351	.9451	.9551	.9651	.9751	.9851	.9950
.850	.9050	.9150	.9250	.9350	.9450	.9550	.9650	.9750	.9850	.9950
.950	.9050	.9150	.9250	.9350	.9450	.9550	.9650	.9750	.9850	.9950

SIGMA Y = .20

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9065	.9164	.9263	.9361	.9460	.9558	.9656	.9755	.9853	.9951
.150	.9065	.9164	.9262	.9361	.9460	.9558	.9656	.9755	.9853	.9951
.250	.9064	.9163	.9262	.9361	.9459	.9558	.9656	.9755	.9853	.9951
.350	.9063	.9162	.9261	.9360	.9459	.9558	.9656	.9755	.9853	.9951
.450	.9062	.9161	.9260	.9359	.9458	.9557	.9656	.9754	.9853	.9951
.550	.9060	.9160	.9259	.9358	.9457	.9557	.9655	.9754	.9853	.9951
.650	.9057	.9157	.9257	.9357	.9456	.9555	.9655	.9754	.9852	.9951
.750	.9054	.9154	.9254	.9354	.9454	.9554	.9653	.9753	.9852	.9951
.850	.9051	.9151	.9252	.9352	.9452	.9552	.9652	.9752	.9851	.9951
.950	.9049	.9150	.9250	.9350	.9450	.9550	.9650	.9750	.9850	.9950

SIGMA Y = .30

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9084	.9181	.9278	.9375	.9471	.9568	.9664	.9760	.9856	.9952
.150	.9083	.9180	.9277	.9374	.9471	.9568	.9664	.9760	.9856	.9952
.250	.9082	.9179	.9277	.9374	.9471	.9567	.9664	.9760	.9856	.9952
.350	.9080	.9178	.9275	.9373	.9470	.9567	.9664	.9760	.9856	.9952
.450	.9077	.9175	.9273	.9371	.9469	.9566	.9663	.9760	.9856	.9952
.550	.9073	.9171	.9270	.9369	.9467	.9565	.9662	.9759	.9856	.9952
.650	.9067	.9166	.9266	.9365	.9464	.9562	.9661	.9758	.9856	.9952
.750	.9060	.9160	.9260	.9360	.9459	.9559	.9658	.9757	.9855	.9952
.850	.9053	.9153	.9253	.9354	.9454	.9554	.9654	.9754	.9853	.9952
.950	.9049	.9149	.9249	.9350	.9450	.9550	.9650	.9750	.9851	.9950

SIGMA Y = .40

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9110	.9205	.9299	.9393	.9488	.9581	.9675	.9768	.9861	.9954
.150	.9108	.9203	.9298	.9393	.9487	.9581	.9675	.9768	.9861	.9954
.250	.9106	.9201	.9297	.9392	.9486	.9580	.9674	.9768	.9861	.9954
.350	.9102	.9199	.9294	.9390	.9485	.9579	.9674	.9767	.9861	.9954
.450	.9097	.9194	.9291	.9387	.9483	.9578	.9673	.9767	.9861	.9954
.550	.9090	.9188	.9286	.9383	.9480	.9576	.9671	.9766	.9860	.9954
.650	.9080	.9179	.9278	.9377	.9474	.9572	.9669	.9765	.9860	.9954
.750	.9068	.9168	.9268	.9367	.9467	.9566	.9664	.9762	.9858	.9953
.850	.9055	.9156	.9256	.9357	.9457	.9557	.9657	.9757	.9856	.9953
.950	.9047	.9148	.9249	.9349	.9450	.9550	.9651	.9751	.9851	.9951

SIGMA Y = .50

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9142	.9234	.9325	.9416	.9507	.9598	.9688	.9778	.9867	.9956
.150	.9139	.9232	.9324	.9415	.9507	.9597	.9688	.9777	.9867	.9956
.250	.9136	.9229	.9322	.9414	.9505	.9596	.9687	.9777	.9867	.9956
.350	.9131	.9225	.9318	.9411	.9503	.9595	.9686	.9777	.9867	.9956
.450	.9124	.9219	.9313	.9407	.9501	.9593	.9685	.9776	.9866	.9956
.550	.9113	.9210	.9306	.9401	.9496	.9590	.9683	.9775	.9866	.9956
.650	.9098	.9196	.9294	.9392	.9488	.9584	.9679	.9773	.9865	.9955
.750	.9076	.9178	.9278	.9378	.9477	.9575	.9672	.9768	.9863	.9955
.850	.9058	.9159	.9260	.9361	.9461	.9562	.9661	.9761	.9859	.9954
.950	.9046	.9147	.9248	.9349	.9450	.9550	.9651	.9751	.9851	.9951

SIGMA Y = .60

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9179	.9268	.9356	.9443	.9530	.9617	.9703	.9788	.9873	.9958
.150	.9176	.9265	.9354	.9442	.9529	.9616	.9702	.9788	.9873	.9958
.250	.9172	.9262	.9351	.9440	.9528	.9615	.9702	.9788	.9873	.9958
.350	.9165	.9256	.9347	.9436	.9525	.9613	.9701	.9787	.9873	.9958
.450	.9155	.9248	.9340	.9431	.9521	.9611	.9699	.9786	.9873	.9958
.550	.9141	.9236	.9330	.9423	.9515	.9606	.9696	.9785	.9872	.9958
.650	.9120	.9217	.9314	.9410	.9505	.9599	.9691	.9782	.9871	.9958
.750	.9092	.9192	.9292	.9391	.9489	.9586	.9682	.9776	.9868	.9957
.850	.9063	.9164	.9266	.9367	.9467	.9567	.9667	.9766	.9863	.9956
.950	.9044	.9145	.9247	.9348	.9449	.9550	.9651	.9752	.9852	.9952

SIGMA Y = .70

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9221	.9306	.9390	.9473	.9556	.9638	.9720	.9800	.9861	.9960
.150	.9217	.9303	.9387	.9471	.9555	.9637	.9719	.9800	.9881	.9960
.250	.9212	.9298	.9384	.9469	.9553	.9636	.9718	.9800	.9880	.9960
.350	.9204	.9292	.9379	.9465	.9550	.9634	.9717	.9799	.9880	.9960
.450	.9192	.9282	.9371	.9458	.9545	.9631	.9715	.9798	.9880	.9960
.550	.9174	.9266	.9356	.9448	.9538	.9625	.9712	.9796	.9879	.9960
.650	.9146	.9243	.9338	.9432	.9525	.9616	.9705	.9793	.9878	.9960
.750	.9109	.9209	.9308	.9407	.9504	.9600	.9694	.9786	.9875	.9960
.850	.9068	.9170	.9272	.9374	.9475	.9575	.9674	.9772	.9867	.9958
.950	.9040	.9143	.9245	.9347	.9449	.9551	.9652	.9753	.9853	.9953

Table B.2b (Cont.)

SIGMA Y = .60

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9266	.9347	.9426	.9505	.9583	.9661	.9737	.9813	.9888	.9963
.150	.9262	.9343	.9424	.9503	.9582	.9660	.9737	.9813	.9888	.9963
.250	.9256	.9339	.9420	.9500	.9580	.9658	.9736	.9812	.9888	.9963
.350	.9247	.9331	.9414	.9496	.9576	.9656	.9734	.9812	.9888	.9963
.450	.9233	.9319	.9405	.9488	.9571	.9652	.9732	.9811	.9887	.9963
.550	.9211	.9301	.9390	.9477	.9562	.9646	.9728	.9808	.9887	.9963
.650	.9178	.9272	.9366	.9457	.9547	.9635	.9721	.9804	.9885	.9963
.750	.9130	.9230	.9329	.9426	.9522	.9616	.9707	.9796	.9882	.9962
.850	.9075	.9178	.9281	.9383	.9484	.9584	.9682	.9779	.9873	.9961
.950	.9036	.9140	.9243	.9346	.9449	.9551	.9653	.9754	.9855	.9954

SIGMA Y = .90

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9314	.9390	.9465	.9538	.9612	.9684	.9756	.9826	.9896	.9966
.150	.9310	.9386	.9462	.9536	.9610	.9683	.9755	.9826	.9896	.9966
.250	.9303	.9381	.9458	.9533	.9608	.9681	.9754	.9826	.9896	.9966
.350	.9293	.9373	.9451	.9528	.9604	.9679	.9752	.9825	.9896	.9966
.450	.9277	.9360	.9441	.9520	.9598	.9675	.9750	.9823	.9895	.9966
.550	.9253	.9339	.9424	.9507	.9589	.9668	.9746	.9821	.9894	.9965
.650	.9214	.9306	.9397	.9485	.9572	.9656	.9738	.9817	.9893	.9965
.750	.9156	.9255	.9353	.9449	.9542	.9634	.9722	.9808	.9889	.9965
.850	.9085	.9189	.9292	.9394	.9495	.9595	.9692	.9788	.9879	.9963
.950	.9031	.9136	.9241	.9345	.9448	.9551	.9654	.9755	.9856	.9955

SIGMA Y = 1.00

PHI X

THETA X	.905	.915	.925	.935	.945	.955	.965	.975	.985	.995
.050	.9363	.9434	.9504	.9572	.9640	.9708	.9774	.9840	.9904	.9968
.150	.9359	.9430	.9501	.9570	.9639	.9707	.9773	.9839	.9904	.9968
.250	.9352	.9425	.9496	.9567	.9637	.9705	.9772	.9839	.9904	.9968
.350	.9341	.9416	.9490	.9562	.9633	.9702	.9771	.9838	.9904	.9968
.450	.9324	.9402	.9479	.9554	.9627	.9698	.9768	.9837	.9903	.9968
.550	.9298	.9380	.9461	.9540	.9617	.9691	.9764	.9834	.9902	.9968
.650	.9254	.9344	.9431	.9516	.9598	.9678	.9756	.9830	.9901	.9968
.750	.9187	.9284	.9380	.9474	.9565	.9654	.9739	.9820	.9897	.9967
.850	.9097	.9202	.9306	.9408	.9509	.9608	.9704	.9798	.9886	.9966
.950	.9023	.9131	.9237	.9343	.9448	.9552	.9655	.9757	.9858	.9957